# FREE BIHOLOMORPHIC FUNCTIONS AND OPERATOR MODEL THEORY, II

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Abstract. In a companion to this paper, we introduced the class of $n$-tuples $f = (f_1, \ldots, f_n)$ of formal power series in noncommutative indeterminates $Z_1, \ldots, Z_n$ with the *model property* and developed an operator model theory for *pure* $n$-tuples of operators in noncommutative domains $\mathbb{B}_f(\mathcal{H}) \subset B(\mathcal{H})^n$, where the associated universal model is an $n$-tuple $(M_{Z_1}, \ldots, M_{Z_n})$ of multiplication operators on a Hilbert space $\mathbb{H}^2(f)$ of formal powers series.

In the present paper, several results concerning the noncommutative multivariable operator theory on the unit ball $[B(\mathcal{H})^n]_1^-$ are extended to noncommutative varieties $\mathcal{V}_{f,J}(\mathcal{H}) \subseteq \mathbb{B}_f(\mathcal{H})$ defined by

$$\mathcal{V}_{f,J}(\mathcal{H}) := \left\{(T_1, \ldots, T_n) \in \mathbb{B}_f(\mathcal{H}) : \ \psi(T_1, \ldots, T_n) = 0 \ \text{ for any } \ \psi \in J\right\},$$

for an appropriate evaluation $\psi(T_1, \ldots, T_n)$, and associated with $n$-tuples $f$ with the model property and WOT-closed two-sided ideals $J$ of the Hardy algebra $H^\infty(\mathbb{B}_f)$, the WOT-closure of all noncommutative polynomials in $M_{Z_1}, \ldots, M_{Z_n}$ and the identity. We develop an operator model theory and dilation theory for $\mathcal{V}_{f,J}(\mathcal{H})$, where the associated universal model is an $n$-tuple $(B_1, \ldots, B_n)$ of operators acting on a Hilbert space $\mathcal{N}_{f,J}$ of formal power series. We study the representations of the algebras generated by $B_1, \ldots, B_n$ and the identity: the variety algebra $\mathcal{A}(\mathcal{V}_{f,J})$, the Hardy algebra $H^\infty(\mathcal{V}_{f,J}))$, and the $C^*$-algebra $C^*(B_1, \ldots, B_n)$. A constrained characteristic function $\Theta_{f,X,J}$, associated with each $n$-tuple $X \in \mathcal{V}_{f,J}(\mathcal{H})$, is used to provide an operator model for the class of completely non-coisometric (c.n.c) elements in the noncommutative variety $\mathcal{V}_{f,J}(\mathcal{H})$. As a consequence, we show that $\Theta_{f,X,J}$ is a complete unitary invariant for the c.n.c. part of $\mathcal{V}_{f,J}(\mathcal{H})$. A Beurling type theorem characterizing the joint invariant subspaces under $B_1, \ldots, B_n$ and a commutant lifting for pure $n$-tuples of operators in $\mathcal{V}_{f,J}(\mathcal{H})$ is also provided. In particular, when $J$ is the $WOT$-closed two-sided ideal generated by the commutators $M_{Z_i}M_{Z_j} - M_{Z_j}M_{Z_i}$, $i, j \in \{1, \ldots, n\}$, we obtain commutative versions for all the results.

For special classes of $n$-tuples of formal power series $f = (f_1, \ldots, f_n)$ and $J = \{0\}$, we obtain several results regarding the dilation and model theory for the noncommutative domain $\mathbb{B}_f(\mathcal{H})$ or the c.n.c. part of it.

## Introduction

This paper is a continuation of [27] in our attempt to to *transfer* the free analogue of Nagy-Foiaş theory (see [28], [6], [5], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [22], [23], [26], [27], [1], [2], [4]) from the closed unit ball

$$[B(\mathcal{H})^n]_1^- := \{(X_1, \ldots, X_n) \in B(\mathcal{H})^n : \ X_1X_1^* + \cdots + X_nX_n^* \leq I\}$$

to other noncommutative domains in $B(\mathcal{H})^n$, where $B(\mathcal{H})$ is the algebra of bounded linear operators an a Hilbert space $\mathcal{H}$. More precisely, we want to find large classes $\mathcal{G}$ of free holomorphic functions $g : \Omega \subseteq [B(\mathcal{H})^n]_1^- \to B(\mathcal{H})^n$ for which a reasonable operator model theory and dilation theory can be developed for the noncommutative domain $g(\Omega)$.

Section 1 contains some preliminaries on the class $\mathcal{M}$ of $n$-tuples of formal power series $f = (f_1, \ldots, f_n)$ with the *model property*. An $n$-tuple $f$ has the model property if it is either one of the following: an $n$-tuple of polynomials with property $(\mathcal{A})$, an $n$-tuple of formal power series with $f(0) = 0$ and property

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$(\mathcal{S})$, or an $n$-tuple of free holomorphic functions with property $(\mathcal{F})$. We associate with each $f \in \mathcal{M}$ a Hilbert space $\mathbb{H}^2(f)$ of formal power series and the noncommutative domain

$$\mathbb{B}_f(\mathcal{H}) := \{X = (X_1, \ldots, X_n) \in B(\mathcal{H})^n : \ g(f(X)) = X \text{ and } \|f(X)\| \leq 1\},$$

where $g = (g_1, \ldots, g_n)$ is the inverse of $f$ with respect to the composition of power series, and the evaluations are well-defined.

The characteristic function of an $n$-tuple $T = (T_1, \ldots, T_n)$ in the noncommutative domain $\mathbb{B}_f(\mathcal{H})$ is the operator $\Theta_{f,T} : \mathbb{H}^2(f) \otimes \mathcal{D}_{f,T^*} \to \mathbb{H}^2(f) \otimes \mathcal{D}_{f,T}$ having the formal Fourier representation

$$-I_{\mathbb{H}^2(f)} \otimes f(T) + \left(I_{\mathbb{H}^2(f)} \otimes \Delta_{f,T}\right) \left(I_{\mathbb{H}^2(f)\otimes \mathcal{H}} - \sum_{i=1}^n \Lambda_i \otimes f_i(T)^*\right)^{-1} [\Lambda_1 \otimes I_{\mathcal{H}}, \ldots, \Lambda_n \otimes I_{\mathcal{H}}] \left(I_{\mathbb{H}^2(f)} \otimes \Delta_{f,T^*}\right),$$

where $\Lambda_1, \ldots, \Lambda_n$ are the right multiplication operators by the power series $f_i$ on the Hardy space $\mathbb{H}^2(f)$ and $\Delta_{f,T}$, $\Delta_{f,T^*}$ are certain defect operators, while $\mathcal{D}_{f,T}$, $\mathcal{D}_{f,T^*}$ are the corresponding defect spaces associated with $T \in \mathbb{B}_f(\mathcal{H})$. We remark that the characteristic function is a contractive multi-analytic operator with respect to the universal model $(M_{Z_1}, \ldots, M_{Z_n})$ associated with the noncommutative domain $\mathbb{B}_f$.

In Section 2, we present operator models for completely non-coisometric (c.n.c.) $n$-tuples of operators $T := (T_1, \ldots, T_n)$ in noncommutative domains $\mathbb{B}_f(\mathcal{H})$, generated by by an $n$-tuples of formal power series $f = (f_1, \ldots, f_n)$ of class $\mathcal{M}^b$, in which the characteristic function $\Theta_{f,T}$ occurs explicitly. More precisely, we show that $T := (T_1, \ldots, T_n)$ is unitarily equivalent to a c.n.c. $n$-tuple $\mathbf{T} := (\mathbf{T}_1, \ldots, \mathbf{T}_n)$ of operators in $\mathbb{B}_f(\mathbf{H})$ on the Hilbert space

$$\mathbf{H} := [(\mathbb{H}^2(f) \otimes \mathcal{D}_{f,T}) \oplus \overline{\Delta_{\Theta_{f,T}}(\mathbb{H}^2(f) \otimes \mathcal{D}_{f,T^*})}] \ominus \{\Theta_{f,T} x \oplus \Delta_{\Theta_{f,T}} x : \quad x \in \mathbb{H}^2(f) \otimes \mathcal{D}_{f,T^*}\},$$

where $\Delta_{\Theta_{f,T}} := (I - \Theta_{f,T}^* \Theta_{f,T})^{1/2}$ and the operator $\mathbf{T}_i$ is defined by

$$\mathbf{T}_i^*[x \oplus \Delta_{\Theta_{f,T}} y] := (M_{Z_i}^* \otimes I_{\mathcal{D}_{f,T}})x \oplus D_i^*(\Delta_{\Theta_{f,T}} y), \qquad i = 1, \ldots, n,$$

for $x \in \mathbb{H}^2(f) \otimes \mathcal{D}_{f,T}$, $y \in \mathbb{H}^2(f) \otimes \mathcal{D}_{f,T^*}$, where $D_i(\Delta_{\Theta_{f,T}} y) := \Delta_{\Theta_{f,T}}(M_{Z_i} \otimes I_{\mathcal{D}_{f,T^*}})y$. Moreover, $T$ is a pure $n$-tuple of operators in $\mathbb{B}_f(\mathcal{H})$ if and only if the characteristic function $\Theta_{f,T}$ is an isometry. In this case, the model reduces to

$$\mathbf{H} = \left(\mathbb{H}^2(f) \otimes \mathcal{D}_{f,T}\right) \ominus \Theta_{f,T}(\mathbb{H}^2(f) \otimes \mathcal{D}_{f,T^*}), \qquad \mathbf{T}_i^* x = (M_{Z_i}^* \otimes I_{\mathcal{D}_{f,T}})x, \qquad x \in \mathbf{H}.$$

This result is used to show that the characteristic function is a complete unitary invariant for the c.n.c. $n$-tuples of operators in $\mathbb{B}_f(\mathcal{H})$. We also show that any contractive multi-analytic operator with respect to $M_{Z_1}, \ldots, M_{Z_n}$ generates a c.n.c. $n$-tuple of operators in $\mathbb{B}_f(\mathbf{H})$, for an appropriate Hilbert space $\mathbf{H}$.

In Section 3, under natural conditions on the $n$-tuple $f = (f_1, \ldots, f_n)$, we study the $*$-representations of the $C^*$-algebra $C^*(M_{Z_1}, \ldots, M_{Z_n})$ and obtain a Wold type decomposition for the nondegenerate $*$-representations, where $(M_{Z_1}, \ldots, M_{Z_n})$ is the universal model associated with the noncommutative domain $\mathbb{B}_f$. We also show that any $n$-tuple $T = (T_1, \ldots, T_n)$ of operators in the noncommutative domain $\mathbb{B}_f(\mathcal{H})$ has a minimal dilation which is unique up to an isomorphism, i.e., there is an $n$-tuple $V := (V_1, \ldots, V_n)$ of operators on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that

(i) $(V_1, \ldots, V_n) \in \mathbb{B}_f(\mathcal{K})$;
(ii) there is a $*$-representation $\pi : C^*(M_{Z_1}, \ldots, M_{Z_n}) \to B(\mathcal{K})$ such that $\pi(M_{Z_i}) = V_i$, $i = 1, \ldots, n$;
(iii) $V_i^*|_{\mathcal{H}} = T_i^*$, $i = 1, \ldots, n$;
(iv) $\mathcal{K} = \bigvee_{\alpha \in \mathbb{F}_n^+} V_\alpha \mathcal{H}$.

A commutant lifting theorem for $\mathbb{B}_f(\mathcal{H})$ (see Theorem 3.8) is also provided.

If $f = (f_1, \ldots, f_n)$ has the model property, we introduce the Hardy algebra $H^\infty(\mathbb{B}_f)$ to be the WOT-closure of all noncommutative polynomials in $M_{Z_1}, \ldots, M_{Z_n}$ and the identity, where $(M_{Z_1}, \ldots, M_{Z_n})$ is the universal model associated with the noncommutative domain $\mathbb{B}_f$. In Section 4, we extend the model theory to c.n.c. $n$-tuples of operators in noncommutative varieties defined by

$$\mathcal{V}_{f,J}(\mathcal{H}) := \{(T_1, \ldots, T_n) \in \mathbb{B}_f(\mathcal{H}) : \ \psi(T_1, \ldots, T_n) = 0 \ \text{ for any } \ \psi \in J\},$$

for an appropriate evaluation $\psi(T_1,\dots,T_n)$, and associated with $n$-tuples $f$ with the model property and WOT-closed two-sided ideals $J$ of the Hardy algebra $H^\infty(\mathbb{B}_f)$. We also show that the constrained characteristic function $\Theta_{f,T,J}$ is a complete unitary invariant for the c.n.c. part of $\mathcal{V}_{f,J}(\mathcal{H})$.

In Section 5, we develop a dilation theory for $n$-tuples of operators $(T_1,\dots,T_n)$ in the noncommutative domain $\mathbb{B}_f(\mathcal{H})$, subject to constraints such as

$$(q\circ f)(T_1,\dots,T_n)=0,\qquad q\in\mathcal{P},$$

where $\mathcal{P}$ is a set of homogeneous noncommutative polynomials. We show that if $f=(f_1,\dots,f_n)$ is an $n$-tuple of formal power series with the radial approximation property and let $B=(B_1,\dots,B_n)$ be the universal model associated with the WOT-closed two-sided ideal $J_{\mathcal{P}\circ f}$ generated by $q(f(M_Z))$, $q\in\mathcal{P}$, in $H^\infty(\mathbb{B}_f)$, then the linear map $\Psi_{f,T,\mathcal{P}}:\overline{\text{span}}\{B_\alpha B_\beta:\ \alpha,\beta\in\mathbb{F}_n^+\}\to B(\mathcal{H})$ defined by

$$\Psi_{f,T,\mathcal{P}}(B_\alpha B_\beta):=T_\alpha T_\beta^*,\qquad \alpha,\beta\in\mathbb{F}_n^+,$$

is completely contractive. If $\mathcal{H}$ is a separable Hilbert space, we prove that there exists a separable Hilbert space $\mathcal{K}_\pi$ and a $*$-representation $\pi: C^*(B_1,\dots,B_n)\to B(\mathcal{K}_\pi)$ which annihilates the compact operators and

$$\sum_{i=1}^n f_i(\pi(B_1),\dots,\pi(B_n))f_i(\pi(B_1),\dots,\pi(B_n))^*=I_{\mathcal{K}_\pi},$$

such that

(a) $\mathcal{H}$ can be identified with a $*$-cyclic co-invariant subspace of $\tilde{\mathcal{K}}:=(\mathcal{N}_{f,J_{\mathcal{P}\circ f}}\otimes\overline{\Delta_{f,T}\mathcal{H}})\oplus\mathcal{K}_\pi$ under the operators

$$V_i:=\begin{bmatrix} B_i\otimes I_{\overline{\Delta_{f,T}\mathcal{H}}} & 0\\ 0 & \pi(B_i)\end{bmatrix},\quad i=1,\dots,n;$$

(b) $T_i^*=V_i^*|\mathcal{H},\ i=1,\dots,n;$

(c) $V:=(V_1,\dots,V_n)\in\mathbb{B}_f(\widetilde{\mathcal{K}})$ and $(q\circ f)(V)=0,\ q\in\mathcal{P}$.

In Section 6, under the conditions that $f=(f_1,\dots,f_n)$ is an $n$-tuple of power series with the model property and $J$ is a WOT-closed two-sided ideal of the Hardy algebra $H^\infty(\mathbb{B}_f)$, we provide a Beurling [3] type theorem characterizing the invariant subspaces under the universal $n$-tuple $(B_1,\dots,B_n)$ associated with a noncommutative variety $\mathcal{V}_{f,J}(\mathcal{H})$, and a commutant lifting theorem [28] for pure $n$-tuples of operators in $\mathcal{V}_{f,J}(\mathcal{H})$.

We remark that all the results of this paper have commutative versions which can be obtained when $J$ is the $WOT$-closed two-sided ideal generated by the commutators $M_{Z_i}M_{Z_j}-M_{Z_j}M_{Z_i}$, $i,j\in\{1,\dots,n\}$. In this case, if $T:=(T_1,\dots,T_n)\in\mathbb{B}_f(\mathcal{H})$ is such that

$$T_iT_j=T_jT_i,\qquad i,j=1,\dots,n,$$

then the characteristic function of $T$ can be identified with the multiplier $M_{\Theta_{f,J,T}}:\mathbb{H}^2(g(\mathbf{B}_n))\otimes\mathcal{D}_{f,T^*}\to\mathbb{H}^2(g(\mathbf{B}_n))\otimes\mathcal{D}_{f,T}$ defined by the operator-valued analytic function

$$\Theta_{f,J,T}(z):=-f(T)+\Delta_{f,T}\left(I-\sum_{i=1}^n f_i(z)f_i(T)^*\right)^{-1}[f_1(z)I_{\mathcal{H}},\dots,f_n(z)I_{\mathcal{H}}]\,\Delta_{f,T^*},\qquad z\in g(\mathbb{B}_n),$$

where $H^2(g(\mathbf{B}_n))$ is a reproducing kernel Hilbert space of holomorphic functions on $g(\mathbf{B}_n)$, $\mathbf{B}_n$ is the open unit ball of $\mathbb{C}^n$, and $g=(g_1,\dots,g_n)$ is the inverse of $f$ with respect to the composition.

It would be interesting to see to what extent the results of this paper and [27] can be extended to the Muhly-Solel setting of tensor algebras over $C^*$-correspondences ([7], [8], [9]).

## 1. Hilbert spaces of formal power series and noncommutative domains

In this section, we recall (see [27]) some basic facts regarding the Hilbert spaces $\mathbb{H}^2(f)$ and the noncommutative domains $\mathbb{B}_f(\mathcal{H})$ associated with $n$-tuples of formal power series $f=(f_1,\ldots,f_n)$ with the *model property.*

Let $\mathbb{F}_n^+$ be the free semigroup with $n$ generators $g_1,\ldots,g_n$ and the identity $g_0$. The length of $\alpha\in\mathbb{F}_n^+$ is defined by $|\alpha|:=0$ if $\alpha=g_0$ and $|\alpha|:=k$ if $\alpha=g_{i_1}\cdots g_{i_k}$, where $i_1,\ldots,i_k\in\{1,\ldots,n\}$. Let $\mathbb{C}[Z_1,\ldots,Z_n]$ be the algebra of noncommutative polynomials with complex coefficients and noncommuting indeterminates $Z_1,\ldots,Z_n$. We say that an $n$-tuple $p=(p_1,\ldots,p_n)$ of polynomials is invertible in $\mathbb{C}[Z_1,\ldots,Z_n]^n$ with respect to the composition if there exists an $n$-tuple $q=(q_1,\ldots,q_n)$ of polynomials such that

$$p\circ q=q\circ p=id.$$

In this case, we say that $p=(p_1,\ldots,p_n)$ has property $(\mathcal{A})$. We introduce an inner product on $\mathbb{C}[Z_1,\ldots,Z_n]$ by setting $\langle p_\alpha,p_\beta\rangle:=\delta_{\alpha\beta}$, $\alpha,\beta\in\mathbb{F}_n^+$, where $p_\alpha:=p_{i_1}\cdots p_{i_k}$ if $\alpha=g_{i_1}\cdots g_{i_k}\in\mathbb{F}_n^+$, and $p_{g_0}:=1$. Let $\mathbb{H}^2(p)$ be the completion of the linear span of the noncommutative polynomials $p_\alpha$, $\alpha\in\mathbb{F}_n^+$, with respect to this inner product.

Denote by $B(\mathcal{H})$ the algebra of all bounded linear operators on an infinite dimensional Hilbert space $\mathcal{H}$ and let $\Omega_0\subset B(\mathcal{H})^n$ be a set containing a ball $[B(\mathcal{H})^n]_r$ for some $r>0$, where

$$[B(\mathcal{H})^n]_r:=\left\{(X_1,\ldots,X_n)\in B(\mathcal{H})^n:\ \|X_1X_1^*+\cdots+X_nX_n^*\|^{1/2}<r\right\}.$$

We say that $f:\Omega_0\to B(\mathcal{H})$ is a free holomorphic function on $\Omega_0$ if there are some complex numbers $a_\alpha$, $\alpha\in\mathbb{F}_n^+$, such that

$$f(X)=\sum_{k=0}^\infty\sum_{|\alpha|=k}a_\alpha X_\alpha,\qquad X=(X_1,\ldots,X_n)\in\Omega_0,$$

where the convergence is in the operator norm topology. Here, we denoted $X_\alpha:=X_{i_1}\cdots X_{i_k}$ if $\alpha=g_{i_1}\cdots g_{i_k}\in\mathbb{F}_n^+$, and $X_{g_0}:=I_\mathcal{H}$. One can show that any free holomorphic function on $\Omega_0$ has a unique representation. The algebra $H_{\mathbf{ball}}$ of free holomorphic functions on the open operatorial $n$-ball of radius one is defined as the set of all formal power series $f=\sum_{\alpha\in\mathbb{F}_n^+}a_\alpha Z_\alpha$ with radius of convergence $r(f)\geq 1$, i.e., $\{a_\alpha\}_{\alpha\in\mathbb{F}_n^+}$ are complex numbers with $r(f)^{-1}:=\limsup_{k\to\infty}\left(\sum_{|\alpha|=k}|a_\alpha|^2\right)^{1/2k}\leq 1$. In this case, the mapping

$$[B(\mathcal{H})^n]_1\ni(X_1,\ldots,X_n)\mapsto f(X_1,\ldots,X_n):=\sum_{k=0}^\infty\sum_{|\alpha|=k}a_\alpha X_\alpha\in B(\mathcal{H})$$

is well-defined, where the convergence is in the operator norm topology. Moreover, the series converges absolutely, i.e., $\sum_{k=0}^\infty\left\|\sum_{|\alpha|=k}a_\alpha X_\alpha\right\|<\infty$ and uniformly on any ball $[B(\mathcal{H})^n]_\gamma$ with $0\leq\gamma<1$. More on free holomorphic functions on the unit ball $[B(\mathcal{H})^n]_1$ can be found in [21], [24], and [25].

The evaluation of $f=\sum_{\alpha\in\mathbb{F}_n^+}a_\alpha Z_\alpha$ is also well-defined if there exists an $n$-tuple $\rho=(\rho_1,\ldots,\rho_n)$ of strictly positive numbers such that

$$\limsup_{k\to\infty}\left(\sum_{|\alpha|=k}|a_\alpha|\rho_\alpha\right)^{1/k}\leq 1.$$

In this case, the series $f(X_1,\ldots,X_n):=\sum_{k=0}^\infty\sum_{|\alpha|=k}a_\alpha X_\alpha$ converges absolutely and uniformly on any noncommutative polydisc

$$P(\mathbf{r}):=\{(X_1,\ldots,X_n)\in B(\mathcal{H})^n:\ \|X_j\|\leq r_j, j=1,\ldots,n\}$$

of multiradius $\mathbf{r}=(r_1,\ldots,r_n)$ with $r_j<\rho_j$, $j=1,\ldots,n$.

We remark that, when $(X_1,\ldots,X_n)\in B(\mathcal{H})^n$ is a nilpotent $n$-tuple of operators, i.e., there is $m\geq 1$ such that $X_\alpha=0$ for all $\alpha\in\mathbb{F}_n^+$ with $|\alpha|=m$, then $f(X_1,\ldots,X_n)$ makes sense for any formal power series $f$.

Let $g = \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha Z_\alpha$ be a formal power series in indeterminates $Z_1, \dots, Z_n$. Denote by $\mathcal{C}_g(\mathcal{H})$ (resp. $\mathcal{C}_g^{SOT}(\mathcal{H})$) the set of all $Y := (Y_1, \dots, Y_n) \in B(\mathcal{H})^n$ such that the series

$$g(Y_1, \dots, Y_n) := \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha Y_\alpha$$

is norm (resp. SOT) convergent. These sets are called sets of norm (resp. SOT) convergence for the power series $g$. We introduce the set $\mathcal{C}_g^{rad}(\mathcal{H})$ of all $Y := (Y_1, \dots, Y_n) \in B(\mathcal{H})^n$ such that there exists $\delta \in (0,1)$ with the property that $rY \in \mathcal{C}_g(\mathcal{H})$ for any $r \in (\delta, 1)$ and

$$\widehat{g}(Y_1, \dots, Y_n) := \text{SOT-}\lim_{r\to 1} \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha r^{|\alpha|} Y_\alpha$$

exists in the strong operator topology. Note that $\mathcal{C}_g(\mathcal{H}) \subseteq \mathcal{C}_g^{SOT}$ and $\mathcal{C}_g^{rad}(\mathcal{H}) \subseteq \overline{\mathcal{C}_g(\mathcal{H})}^{SOT}$.

Consider an $n$-tuple of formal power series $f = (f_1, \dots, f_n)$ in indeterminates $Z_1, \dots, Z_n$ with the Jacobian

$$\det J_f(0) := \det \left[ \left.\frac{\partial f_i}{\partial Z_j}\right|_{Z=0} \right]_{i,j=1}^n \neq 0.$$

As shown in [27], the set $\{f_\alpha\}_{\alpha \in \mathbb{F}_n^+}$ is linearly independent in $\mathbf{S}[Z_1, \dots, Z_n]$, the algebra of all formal powers series in $Z_1, \dots, Z_n$. We introduce an inner product on the linear span of $\{f_\alpha\}_{\alpha \in \mathbb{F}_n^+}$ by setting

$$\langle f_\alpha, f_\beta \rangle := \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases} \quad \text{for} \quad \alpha, \beta \in \mathbb{F}_n^+.$$

Let $\mathbb{H}^2(f)$ be the completion of the linear span of $\{f_\alpha\}_{\alpha \in \mathbb{F}_n^+}$ with respect to this inner product. Assume now that $f(0) = 0$. As seen in [27], $f$ is not a *right zero divisor* with respect to the composition of power series, i.e., there is no non-zero formal power series $G \in \mathbf{S}[Z_1, \dots, Z_n]$ such that $G \circ f = 0$. Consequently, the elements of $\mathbb{H}^2(f)$ can be seen as formal power series in $\mathbf{S}[Z_1, \dots, Z_n]$ of the form $\sum_{\alpha \in \mathbb{F}_n^+} a_\alpha f_\alpha$, where $\sum_{\alpha \in \mathbb{F}_n^+} |a_\alpha|^2 < \infty$.

Let $f = (f_1, \dots, f_n)$ be an $n$-tuple of formal power series in $Z_1, \dots, Z_n$ such that $f(0) = 0$. We say that $f$ has property $(\mathcal{S})$ if the following conditions hold.

- $(\mathcal{S}_1)$ The $n$-tuple $f$ has nonzero radius of convergence and $\det J_f(0) \neq 0$.
- $(\mathcal{S}_2)$ The indeterminates $Z_1, \dots, Z_n$ are in the Hilbert space $\mathbb{H}^2(f)$ and each left multiplication operator $M_{Z_i} : \mathbb{H}^2(f) \to \mathbb{H}^2(f)$ defined by
$$M_{Z_i}\psi := Z_i \psi, \qquad \psi \in \mathbb{H}^2(f),$$
is a bounded multiplier of $\mathbb{H}^2(f)$.
- $(\mathcal{S}_3)$ The left multiplication operators $M_{f_j} : \mathbb{H}^2(f) \to \mathbb{H}^2(f)$, $M_{f_j}\psi = f_j \psi$, satisfy the equations
$$M_{f_j} = f_j(M_{Z_1}, \dots, M_{Z_n}), \qquad j = 1, \dots, n,$$
where $(M_{Z_1}, \dots, M_{Z_n})$ is either in the convergence set $\mathcal{C}_f^{SOT}(\mathbb{H}^2(f))$ or $\mathcal{C}_f^{rad}(\mathbb{H}^2(f))$.

Note that if $f$ is an $n$-tuple of noncommutative polynomials, then the condition $(\mathcal{S}_3)$ is always satisfied. We remark that, when $(M_{Z_1}, \dots, M_{Z_n})$ is in the set $\mathcal{C}_f^{rad}(\mathbb{H}^2(f))$, then the condition $(\mathcal{S}_3)$ should be understood as

$$M_{f_j} = \widehat{f}_j(M_{Z_1}, \dots, M_{Z_n}) := \text{SOT-}\lim_{r\to 1} f_j(rM_{Z_1}, \dots, rM_{Z_n}), \qquad j = 1, \dots, n.$$

Now, we introduce the class of $n$-tuples of free holomorphic function with property $(\mathcal{F})$. Let $\varphi = (\varphi_1, \dots, \varphi_n)$ be an $n$-tuple of free holomorphic functions on $[B(\mathcal{H})^n]_\gamma$, $\gamma > 0$, with range in $[B(\mathcal{H})^n]_1$ and such that $\varphi$ is not a right zero divisor with respect to the composition with free holomorphic functions on $[B(\mathcal{H})^n]_1$. Consider the Hilbert space of free holomorphic functions

$$\mathbb{H}^2(\varphi) := \{G \circ \varphi : \ G \in H^2_{\mathbf{ball}}\},$$

with the inner product

$$\langle F\circ\varphi, G\circ\varphi\rangle_{\mathbb{H}^2(\varphi)} := \langle F, G\rangle_{H^2_{\mathbf{ball}}}.$$

We recall that the noncommutative Hardy space $H^2_{\mathbf{ball}}$ is the Hilbert space of all free holomorphic functions on $[B(\mathcal{H})^n]_1$ of the form

$$f(X_1,\ldots,X_n)=\sum_{k=0}^\infty\sum_{|\alpha|=k} a_\alpha X_\alpha, \qquad \sum_{\alpha\in\mathbb{F}_n^+}|a_\alpha|^2<\infty,$$

with the inner product $\langle f,g\rangle := \sum_{k=0}^\infty\sum_{|\alpha|=k} a_\alpha \overline{b}_\alpha$, where $g=\sum_{k=0}^\infty\sum_{|\alpha|=k} b_\alpha X_\alpha$ is another free holomorphic function in $H^2_{\mathbf{ball}}$. We say that $\varphi$ has property $(\mathcal{F})$ if the following conditions hold.

- $(\mathcal{F}_1)$ The $n$-tuple $\varphi=(\varphi_1,\ldots,\varphi_n)$ has the range in $[B(\mathcal{H})^n]_1$ and it is not a right zero divisor with respect to the composition with free holomorphic functions on $[B(\mathcal{H})^n]_1$.
- $(\mathcal{F}_2)$ The coordinate functions $X_1,\ldots,X_n$ on $[B(\mathcal{H})^n]_\gamma$ are contained in $\mathbb{H}^2(\varphi)$ and the left multiplication by $X_i$ is a bounded multiplier of $\mathbb{H}^2(\varphi)$.
- $(\mathcal{F}_3)$ For each $i=1,\ldots,n$, the left multiplication operator $M_{\varphi_i}:\mathbb{H}^2(\varphi)\to\mathbb{H}^2(\varphi)$ satisfies the equation

$$M_{\varphi_i}=\varphi_i(M_{Z_1},\ldots,M_{Z_n}),$$

  where $(M_{Z_1},\ldots,M_{Z_n})$ is either in the convergence set $\mathcal{C}^{SOT}_\varphi(\mathbb{H}^2(\varphi))$ or $\mathcal{C}^{rad}_\varphi(\mathbb{H}^2(\varphi))$.

We mention that if $\varphi$ is an $n$-tuple of noncommutative polynomials, then the condition $(\mathcal{F}_3)$ is always satisfied.

An $n$-tuple $f=(f_1,\ldots,f_n)$ of formal power series is said to have the *model property* if it is either one of the following:

- (i) an $n$-tuple of polynomials with property $(\mathcal{A})$;
- (ii) an $n$-tuple of formal power series with $f(0)=0$ and property $(\mathcal{S})$;
- (iii) an $n$-tuple of free holomorphic functions with property $(\mathcal{F})$.

We denote by $\mathcal{M}$ the set of all $n$-tuples $f$ with the model property. For several examples of formal power series with the model property we refer the reader to [27].

Let $f=(f_1,\ldots,f_n)$ have the model property and let $g=(g_1,\ldots,g_n)$ be the $n$-tuple of power series having the representations

$$g_i:=\sum_{k=0}^\infty\sum_{\alpha\in\mathbb{F}_n^+,|\alpha|=k} a_\alpha^{(i)} Z_\alpha, \qquad i=1,\ldots,n,$$

where the sequence of complex numbers $\{a_\alpha^{(i)}\}_{\alpha\in\mathbb{F}_n^+}$ is uniquely defined by the condition $g\circ f=id$. We say that an $n$-tuple of operators $X=(X_1,\ldots,X_n)\in B(\mathcal{H})^n$ satisfies the equation $g(f(X))=X$ if either one of the following conditions holds:

(a) $X\in\mathcal{C}^{SOT}_f(\mathcal{H})$ and either

$$X_i=\sum_{k=0}^\infty\sum_{\alpha\in\mathbb{F}_n^+,|\alpha|=k} a_\alpha^{(i)}[f(X)]_\alpha, \qquad i=1,\ldots,n,$$

where the convergence of the series is in the strong operator topology, or

$$X_i=\text{SOT-}\lim_{r\to1}\sum_{k=0}^\infty\sum_{\alpha\in\mathbb{F}_n^+,|\alpha|=k} a_\alpha^{(i)} r^{|\alpha|}[f(X)]_\alpha, \qquad i=1,\ldots,n;$$

(b) $X\in\mathcal{C}^{rad}_f(\mathcal{H})$ and either

$$X_i=\sum_{k=0}^\infty\sum_{\alpha\in\mathbb{F}_n^+,|\alpha|=k} a_\alpha^{(i)}[\widehat{f}(X)]_\alpha, \qquad i=1,\ldots,n,$$

where the convergence of the series is in the strong operator topology, or

$$X_i = \text{SOT-}\lim_{r\to 1}\sum_{k=0}^{\infty}\sum_{\alpha\in\mathbb{F}_n^+,|\alpha|=k} a_\alpha^{(i)} r^{|\alpha|}[\widehat{f}(X)]_\alpha, \qquad i=1,\dots,n.$$

We define the noncommutative domain associated with $f$ by setting

$$\mathbb{B}_f(\mathcal{H}) := \{X=(X_1,\dots,X_n)\in B(\mathcal{H})^n:\ g(f(X))=X \text{ and } \|f(X)\|\le 1\},$$

where $g=(g_1,\dots,g_n)$ is the inverse of $f$ with respect to the composition of power series, and the evaluations are well-defined as above. Note that the condition $g(f(X))=X$ is automatically satisfied when $f$ is an $n$-tuple of polynomials with property $(\mathcal{A})$.

## 2. Characteristic functions and models for $n$-tuples of operators in $\mathbb{B}_f^{cnc}(\mathcal{H})$

In this section, we present models for completely non-coisometric (c.n.c.) $n$-tuples of operators in noncommutative domains $\mathbb{B}_f(\mathcal{H})$, generated by $n$-tuples of formal power series $f=(f_1,\dots,f_n)$ of class $\mathcal{M}^b$, in which the characteristic function occurs explicitly. This is used to show that the characteristic function is a complete unitary invariant for the c.n.c. $n$-tuples of operators in $\mathbb{B}_f(\mathcal{H})$. We also show that any contractive multi-analytic operator with respect to $M_{Z_1},\dots,M_{Z_n}$ generates a c.n.c. $n$-tuple of operators $\mathbf{T}:=(\mathbf{T}_1,\dots,\mathbf{T}_n)\in\mathbb{B}_f(\mathbf{H})$.

First, we need a few definitions. Let $f=(f_1,\dots,f_n)$ be an $n$-tuple of formal power series with the model property. We say that $f$ has the radial approximation property, and write $f\in\mathcal{M}_{rad}$, if there is $\delta\in(0,1)$ such that $(rf_1,\dots,rf_n)$ has the model property for any $r\in(\delta,1]$. Denote by $\mathcal{M}^{||}$ the set of all formal power series $f=(f_1,\dots,f_n)$ having the model property and such that the universal model $(M_{Z_1},\dots,M_{Z_n})$ associated with the noncommutative domain $\mathbb{B}_f$ is in the set of norm-convergence (or radial norm-convergence) of $f$. We also introduce the class $\mathcal{M}_{rad}^{||}$ of all formal power series $f=(f_1,\dots,f_n)$ with the property that there is $\delta\in(0,1)$ such that $rf\in\mathcal{M}^{||}$ for any $r\in(\delta,1]$. We recall that in all the examples presented in [27], the corresponding $n$-tuples $f=(f_1,\dots,f_n)$ are in the class $\mathcal{M}_{rad}^{||}$. Moreover, the $n$-tuples of polynomials with property $(\mathcal{A})$ are also in the class $\mathcal{M}_{rad}^{||}$.

Let $f=(f_1,\dots,f_n)$ be an $n$-tuple of formal power series with the model property and assume that $f_i$ has the representation $f_i(Z_1,\dots,Z_n)=\sum_{\alpha\in\mathbb{F}_n^+} a_\alpha^{(i)} Z_\alpha$. We say that $f$ is in the class $\mathcal{M}^b$ if either one of the following conditions holds:

(i) the $n$-tuple $(M_{Z_1},\dots,M_{Z_n})$ is in the convergence set $\mathcal{C}_f^{SOT}(\mathbb{H}^2(f))$ and

$$\sup_{m\in\mathbb{N}}\left\|\sum_{|\alpha|\le m} a_\alpha^{(i)} M_{Z_\alpha}\right\|<\infty, \qquad i=1,\dots,n;$$

(ii) the $n$-tuple $(M_{Z_1},\dots,M_{Z_n})$ is in the convergence set $\mathcal{C}_f^{rad}(\mathbb{H}^2(f))$ and

$$\sup_{r\in[0,1)}\left\|\sum_{k=0}^{\infty}\sum_{|\alpha|=k} a_\alpha^{(i)} r^{|\alpha|} M_{Z_\alpha}\right\|<\infty, \qquad i=1,\dots,n.$$

Note that $\mathcal{M}_{rad}^{||}\subset\mathcal{M}^{||}\subset\mathcal{M}^b\subset\mathcal{M}$.

We recall that the noncommutative domain associated with $f\in\mathcal{M}$ is

$$\mathbb{B}_f(\mathcal{H}) := \{X=(X_1,\dots,X_n)\in B(\mathcal{H})^n:\ g(f(X))=X \text{ and } \|f(X)\|\le 1\},$$

where $g$ is the inverse power series of $f$ with respect to the composition. We say that $T:=(T_1,\dots,T_n)\in B(\mathcal{H})^n$ is a pure $n$-tuple of operators in $\mathbb{B}_f(\mathcal{H})$ if

$$\text{SOT-}\lim_{k\to\infty}\sum_{\alpha\in\mathbb{F}_n,\,|\alpha|=k}[f(T)]_\alpha[f(T)]_\alpha^*=0.$$

The set of all pure elements of $\mathbb{B}_f(\mathcal{H})$ is denoted by $\mathbb{B}_f^{pure}(\mathcal{H})$. An $n$-tuple $T \in \mathbb{B}_f(\mathcal{H})$ is called *completely non-coisometric* (c.n.c) if there is no vector $h \in \mathcal{H}$, $h \neq 0$, such that

$$\left\langle \Phi_{f,T}^m(I)h, h\right\rangle = \|h\|^2 \quad \text{for any} \quad m = 1, 2, \ldots,$$

where the positive linear mapping $\Phi_{f,T} : B(\mathcal{H} \to B(\mathcal{H})$ is defined by $\Phi_{f,T}(Y) := \sum_{i=1}^n f_i(T)Yf_i(T)^*$. The set of all c.n.c. elements of $\mathbb{B}_f(\mathcal{H})$ is denoted by $\mathbb{B}_f^{cnc}(\mathcal{H})$. Note that

$$\mathbb{B}_f^{pure}(\mathcal{H}) \subseteq \mathbb{B}_f^{cnc}(\mathcal{H}) \subseteq \mathbb{B}_f(\mathcal{H}).$$

Let $H_n$ be an $n$-dimensional complex Hilbert space with orthonormal basis $e_1, e_2, \ldots, e_n$, where $n \in \{1, 2, \ldots\}$. We consider the full Fock space of $H_n$ defined by

$$F^2(H_n) := \mathbb{C}1 \oplus \bigoplus_{k \geq 1} H_n^{\otimes k},$$

where $H_n^{\otimes k}$ is the (Hilbert) tensor product of $k$ copies of $H_n$. Define the left (resp. right) creation operators $S_i$ (resp. $R_i$), $i = 1, \ldots, n$, acting on $F^2(H_n)$ by setting

$$S_i\varphi := e_i \otimes \varphi, \qquad \varphi \in F^2(H_n),$$

(resp. $R_i\varphi := \varphi \otimes e_i$). The noncommutative disc algebra $\mathcal{A}_n$ (resp. $\mathcal{R}_n$) is the norm closed algebra generated by the left (resp. right) creation operators and the identity. The noncommutative analytic Toeplitz algebra $F_n^\infty$ (resp. $R_n^\infty$) is the the weakly closed version of $\mathcal{A}_n$ (resp. $\mathcal{R}_n$). These algebras were introduced in [12] in connection with a noncommutative version of the classical von Neumann inequality [29].

A free holomorphic function $g$ on $[B(\mathcal{H})^n]_1$ is bounded if $\|g\|_\infty := \sup \|g(X)\| < \infty$, where the supremum is taken over all $X \in [B(\mathcal{H})^n]_1$ and $\mathcal{H}$ is an infinite dimensional Hilbert space. Let $H_{\mathbf{ball}}^\infty$ be the set of all bounded free holomorphic functions and let $A_{\mathbf{ball}}$ be the set of all elements $f \in H_{\mathbf{ball}}^\infty$ such that the mapping

$$[B(\mathcal{H})^n]_1 \ni (X_1, \ldots, X_n) \mapsto g(X_1, \ldots, X_n) \in B(\mathcal{H})$$

has a continuous extension to the closed ball $[B(\mathcal{H})^n]_1^-$. We showed in [21] that $H_{\mathbf{ball}}^\infty$ and $A_{\mathbf{ball}}$ are Banach algebras under pointwise multiplication and the norm $\|\cdot\|_\infty$. The noncommutative Hardy space $H_{\mathbf{ball}}^\infty$ can be identified to the noncommutative analytic Toeplitz algebra $F_n^\infty$ . More precisely, a bounded free holomorphic function $g$ is uniquely determined by its *(model) boundary function* $\widetilde{g} \in F_n^\infty$ defined by $\widetilde{g} := \text{SOT-}\lim_{r\to 1} g(rS_1, \ldots, rS_n)$. Moreover, $g$ is the noncommutative Poisson transform [17] of $\widetilde{g}$ at $X \in [B(\mathcal{H})^n]_1$, i.e., $g(X) = P_X[\widetilde{g} \otimes I]$. Similar results hold for bounded free holomorphic functions on the noncommutative ball $[B(\mathcal{H})^n]_\gamma$, $\gamma > 0$.

The next result provides a characterization for the $n$-tuples of formal power series $f$ with property $(\mathcal{S})$ which are in $\mathcal{M}^b$. A similar result holds if $f$ has property $(\mathcal{F})$.

**Lemma 2.1.** *Let $f = (f_1, \ldots, f_n)$ be an $n$-tuple of formal power series with $f(0) = 0$ and $\det J_f(0) \neq 0$. Assume that $f_i$ has the representation $f_i(Z_1, \ldots, Z_n) = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha^{(i)} Z_\alpha$ and let $g = (g_1, \ldots, g_n)$ be the inverse of $f$ under the composition. Then $f$ is in the class $\mathcal{M}^b$ if and only if each $g_i$ is a bounded free holomorphic function on $[B(\mathcal{H})^n]_1$ and either one of the following conditions holds:*

(i) $S_i = \text{SOT-}\lim_{m\to\infty} \sum_{|\alpha| \leq m} a_\alpha^{(i)} \widetilde{g}_\alpha$ *and*

$$\sup_{m \in \mathbb{N}} \left\| \sum_{|\alpha| \leq m} a_\alpha^{(i)} \widetilde{g}_\alpha \right\| < \infty, \qquad i = 1, \ldots, n;$$

(ii) $S_i = \text{SOT-}\lim_{r\to 1} \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha^{(i)} r^{|\alpha|} \widetilde{g}_\alpha$ *and*

$$\sup_{r \in [0,1)} \left\| \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha^{(i)} r^{|\alpha|} \widetilde{g}_\alpha \right\| < \infty, \qquad i = 1, \ldots, n,$$

*where the series converges in the operator norm topology.*

*Proof.* Assume that $f$ has the property $(\mathcal{S})$ and let $g=(g_1,\dots,g_n)$ be its inverse with respect to the composition. Let $U:\mathbb{H}^2(f)\to F^2(H_n)$ be the unitary operator defined by $U(f_\alpha):=e_\alpha$, $\alpha\in\mathbb{F}_n^+$. Note that $Z_i=\sum_{\alpha\in\mathbb{F}_n^+} b_\alpha^{(i)} f_\alpha = U^{-1}(\varphi_i)$ for some coefficients $b_\alpha^{(i)}$ such that $\varphi:=\sum_{\alpha\in\mathbb{F}_n^+} b_\alpha^{(i)} e_\alpha\in F^2(H_n)$. Note that $M_{Z_i}$ is a bounded left multiplier of $\mathbb{H}^2(f)$ if and only if $\varphi_i$ is a bounded left multiplier of $F^2(H_n)$. Moreover, $M_{Z_i}=U^{-1}\varphi_i(S_1,\dots,S_n)U$, where $\varphi_i(S_1,\dots,S_n)$ is in the noncommutative Hardy algebra $F_n^\infty$ and has the Fourier representation $\sum_{\alpha\in\mathbb{F}_n^+} b_\alpha^{(i)} S_\alpha$. According to Theorem 3.1 from [21], we deduce that $g_i=\sum_{\alpha\in\mathbb{F}_n^+} a_\alpha^{(i)} Z_\alpha$ is a bounded free holomorphic function on the unit ball $[B(\mathcal{H})^n]_1$ and has its model boundary function $\widetilde{g}_i=\varphi_i(S_1,\dots,S_n)$. On the other hand, note that the left multiplication operator $M_{f_j}:\mathbb{H}^2(f)\to\mathbb{H}^2(f)$ defined by

$$M_{f_j}\left(\sum_{\alpha\in\mathbb{F}_n^+} c_\alpha f_\alpha\right)=\sum_{\alpha\in\mathbb{F}_n^+} c_\alpha f_j f_\alpha,\qquad \sum_{\alpha\in\mathbb{F}_n^+}|c_\alpha|^2<\infty,$$

satisfies the equation

$$M_{f_j}=U^{-1}S_jU,\qquad j=1,\dots,n,$$

where $S_1,\dots,S_n$ are the left creation operators on $F^2(H_n)$. Since $M_{Z_i}=U^{-1}\widetilde{g}_iU$, where $\widetilde{g}_i$ is the model boundary function of $g_i\in H^\infty_{\mathbf{ball}}$, it is easy to see that the relation $M_{f_j}=f_j(M_{Z_1},\dots,M_{Z_n})$ for $j=1,\dots,n$ is equivalent to the fact that the model boundary function $\widetilde{g}=(\widetilde{g}_1,\dots,\widetilde{g}_n)$ satisfies either one of the following conditions:

(a) $\widetilde{g}$ is in $\mathcal{C}_f^{SOT}(\mathbb{H}^2(f))$ and $S_i=f_i(\widetilde{g}_1,\dots,\widetilde{g}_n)$, $i=1,\dots,n$;
(b) $\widetilde{g}$ is in $\mathcal{C}_f^{rad}(\mathbb{H}^2(f))$ and $S_i=\text{SOT-}\lim_{r\to 1} f_j(r\widetilde{g}_1,\dots,r\widetilde{g}_n)$ for $i=1,\dots,n$.

Now, one can easily see that $f\in\mathcal{M}^b$ if and only if the conditions in the lemma hold. $\square$

**Proposition 2.2.** *Let $f=(f_1,\dots,f_n)$ be an $n$-tuple of formal power series in the class $\mathcal{M}^b$ and let $g=(g_1,\dots,g_n)$ be its inverse under the composition. Then the set $\mathbb{B}_f^{cnc}(\mathcal{H})$ coincides with the image of all c.n.c. row contractions under $g$. Moreover, $\mathbb{B}_f^{pure}(\mathcal{H})=g([B(\mathcal{H})^n]_1^{pure})$.*

*Proof.* Set $[B(\mathcal{H})^n]_1^{cnc}:=\{X\in[B(\mathcal{H})^n]_1^-:\ X \text{ is a c.n.c. row contraction}\}$ and note that $\mathbb{B}_f^{cnc}(\mathcal{H})\subseteq\{g(Y):\ Y\in[B(\mathcal{H})^n]_1^{cnc}\}$. To prove the reversed inclusion let $W=g(Y)$, where $Y\in[B(\mathcal{H})^n]_1^{cnc}$. Assume that $f_i$ has the representation $f_i(Z_1,\dots,Z_n)=\sum_{\alpha\in\mathbb{F}_n^+} a_\alpha^{(i)}Z_\alpha$. Since $f\in\mathcal{M}^b$, using Lemma 2.1 and the fact that the noncommutative Poisson transform $P_Y$ is $SOT$-continuous on bounded sets (since $Y$ is a c.n.c. row contraction), we deduce that either

$$\begin{aligned}Y_i=P_Y[S_i\otimes I]&=\text{SOT-}\lim_{m\to\infty}\sum_{|\alpha|\leq m} a_\alpha^{(i)}P_Y[\widetilde{g}_\alpha\otimes I]\\&=\text{SOT-}\lim_{m\to\infty}\sum_{|\alpha|\leq m} a_\alpha^{(i)}[g(Y)]_\alpha\end{aligned}$$

or

$$\begin{aligned}Y_i=P_Y[S_i\otimes I]&=\text{SOT-}\lim_{r\to 1}\sum_{k=0}^\infty\sum_{|\alpha|=k} a_\alpha^{(i)}r^{|\alpha|}P_Y[\widetilde{g}_\alpha\otimes I]\\&=\text{SOT-}\lim_{r\to 1}\sum_{k=0}^\infty\sum_{|\alpha|=k} a_\alpha^{(i)}r^{|\alpha|}[g(Y)]_\alpha.\end{aligned}$$

Therefore, we obtain $Y_i=f_i(g_1(Y),\dots,g_n(Y))$ for $i=1,\dots,n$. Consequently, we have $f(g(Y))=Y$ which implies that $f(W)=f(g(Y))=Y$ and $g(f(W))=g(Y)=W$, and shows that $W\in\mathbb{B}_f^{cnc}(\mathcal{H})$. Therefore, $\mathbb{B}_f^{cnc}(\mathcal{H})=g([B(\mathcal{H})^n]_1^{cnc})$, the function $g$ is one-to-one on $[B(\mathcal{H})^n]_1^{cnc}$, and $f$ is its inverse on $\mathbb{B}_f^{cnc}(\mathcal{H})$. Consequently, since $\mathbb{B}_f^{pure}(\mathcal{H})\subset\mathbb{B}_f^{cnc}(\mathcal{H})$, we deduce that $\mathbb{B}_f^{pure}(\mathcal{H})=g([B(\mathcal{H})^n]_1^{pure})$. The proof is complete. $\square$

For simplicity, throughout this paper, $T := [T_1, \dots, T_n]$ denotes either the $n$-tuple $(T_1, \dots, T_n)$ of bounded linear operators on a Hilbert space $\mathcal{H}$ or the row operator matrix $[T_1 \ \cdots \ T_n]$ acting from $\mathcal{H}^{(n)}$ to $\mathcal{H}$, where $\mathcal{H}^{(n)} := \oplus_{i=1}^n \mathcal{H}$ is the direct sum of $n$ copies of $\mathcal{H}$. Assume that $T := [T_1, \dots, T_n]$ is a row contraction, i.e.,

$$T_1T_1^* + \cdots + T_nT_n^* \leq I.$$

The defect operators of $T$ are

$$\Delta_T := \left( I_{\mathcal{H}} - \sum_{i=1}^n T_iT_i^* \right)^{1/2} \in B(\mathcal{H}) \quad \text{and} \quad \Delta_{T^*} := (I_{\mathcal{H}^{(n)}} - T^*T)^{1/2} \in B(\mathcal{H}^{(n)}),$$

and the defect spaces of $T$ are defined by

$$\mathcal{D}_T := \overline{\Delta_T\mathcal{H}} \quad \text{and} \quad \mathcal{D}_{T^*} := \overline{\Delta_{T^*}\mathcal{H}^{(n)}}.$$

We recall that the characteristic function of a row contraction $T := [T_1, \dots, T_n]$ is the multi-analytic operator $\Theta_T : F^2(H_n) \otimes \mathcal{D}_{T^*} \to F^2(H_n) \otimes \mathcal{D}_T$ with the formal Fourier representation

$$-I \otimes T + (I \otimes \Delta_T) \left( I - \sum_{i=1}^n R_i \otimes T_i^* \right)^{-1} [R_1 \otimes I_{\mathcal{H}}, \dots, R_n \otimes I_{\mathcal{H}}] \left( I \otimes \Delta_{T^*} \right),$$

where $R_1, \dots, R_n$ are the right creation operators on the full Fock space $F^2(H_n)$. The characteristic function associated with an arbitrary row contraction $T := [T_1, \dots, T_n]$, $T_i \in B(\mathcal{H})$, was introduce in [11] (see [28] for the classical case $n = 1$) and it was proved to be a complete unitary invariant for completely non-coisometric (c.n.c.) row contractions.

Now, let $f = (f_1, \dots, f_n)$ be an $n$-tuple of formal power series with the model property. The characteristic function of an $n$-tuple $T = (T_1, \dots, T_n) \in \mathbb{B}_f(\mathcal{H})$ was introduced in [27] as the multi-analytic operator with respect to $M_{Z_1}, \dots, M_{Z_n}$,

$$\Theta_{f,T} : \mathbb{H}^2(f) \otimes \mathcal{D}_{f,T^*} \to \mathbb{H}^2(f) \otimes \mathcal{D}_{f,T},$$

with the formal Fourier representation

$$\begin{aligned} -I_{\mathbb{H}^2(f)} \otimes f(T) + \left( I_{\mathbb{H}^2(f)} \otimes \Delta_{f,T} \right) & \left( I_{\mathbb{H}^2(f) \otimes \mathcal{H}} - \sum_{i=1}^n \Lambda_i \otimes f_i(T)^* \right)^{-1} \\ & [\Lambda_1 \otimes I_{\mathcal{H}}, \dots, \Lambda_n \otimes I_{\mathcal{H}}] \left( I_{\mathbb{H}^2(f)} \otimes \Delta_{f,T^*} \right), \end{aligned}$$

where $\Lambda_1, \dots, \Lambda_n$ are the right multiplication operators by the power series $f_i$ on the Hardy space $\mathbb{H}^2(f)$ and the defect operators associated with $T := (T_1, \dots, T_n) \in \mathbb{B}_f(\mathcal{H})$ are

$$\Delta_{f,T} := \left( I_{\mathcal{H}} - \sum_{i=1}^n f_i(T)f_i(T)^* \right)^{1/2} \in B(\mathcal{H}) \quad \text{and} \quad \Delta_{f,T^*} := (I - f(T)^*f(T))^{1/2} \in B(\mathcal{H}^{(n)}),$$

while the defect spaces are $\mathcal{D}_{f,T} := \overline{\Delta_{f,T}\mathcal{H}}$ and $\mathcal{D}_{f,T^*} := \overline{\Delta_{f,T^*}\mathcal{H}^{(n)}}$. We recall that a bounded operator $\Phi : \mathbb{H}^2(f) \otimes \mathcal{K}_1 \to \mathbb{H}^2(f) \otimes \mathcal{K}_2$ is multi-analytic with respect to $M_{Z_1}, \dots, M_{Z_n}$ if $\Phi(M_{Z_i} \otimes I_{\mathcal{K}_1}) = (M_{Z_i} \otimes I_{\mathcal{K}_2})\Phi$ for any $i = 1, \dots, n$.

In what follows, we present a model for the $n$-tuples of operators in $\mathbb{B}_f^{cnc}(\mathcal{H})$ in which the characteristic function occurs explicitly.

**Theorem 2.3.** *Let $f = (f_1, \dots, f_n)$ be an $n$-tuple of formal power series in the class $\mathcal{M}^b$ and let $(M_{Z_1}, \dots, M_{Z_n})$ be the universal model associated with the noncommutative domain $\mathbb{B}_f$. Every $n$-tuple of operators $T := (T_1, \dots, T_n)$ in $\mathbb{B}_f^{cnc}(\mathcal{H})$ is unitarily equivalent to an $n$-tuple $\mathbf{T} := (\mathbf{T}_1, \dots, \mathbf{T}_n)$ in $\mathbb{B}_f^{cnc}(\mathbf{H})$ on the Hilbert space*

$$\mathbf{H} := [(\mathbb{H}^2(f) \otimes \mathcal{D}_{f,T}) \oplus \overline{\Delta_{\Theta_{f,T}}(\mathbb{H}^2(f) \otimes \mathcal{D}_{f,T^*})}] \ominus \{\Theta_{f,T}x \oplus \Delta_{\Theta_{f,T}}x : \quad x \in \mathbb{H}^2(f) \otimes \mathcal{D}_{f,T^*}\},$$

*where* $\Delta_{\Theta_{f,T}} := (I - \Theta_{f,T}^*\Theta_{f,T})^{1/2}$ *and the operator $\mathbf{T}_i$ is defined by*

$$\mathbf{T}_i^*[x \oplus \Delta_{\Theta_{f,T}}y] := (M_{Z_i}^* \otimes I_{\mathcal{D}_{f,T}})x \oplus D_i^*(\Delta_{\Theta_{f,T}}y), \qquad i = 1, \dots, n,$$

*for* $x \in \mathbb{H}^2(f) \otimes \mathcal{D}_{f,T}$, $y \in \mathbb{H}^2(f) \otimes \mathcal{D}_{f,T^*}$, *where* $D_i(\Delta_{\Theta_{f,T}} y) := \Delta_{\Theta_{f,T}}(M_{Z_i} \otimes I_{\mathcal{D}_{f,T^*}})y$.

*Moreover,* $T$ *is a pure* $n$*-tuple of operators in* $\mathbb{B}_f(\mathcal{H})$ *if and only if the characteristic function* $\Theta_{f,T}$ *is an isometry. In this case the model reduces to*

$$\mathbf{H} = \left(\mathbb{H}^2(f) \otimes \mathcal{D}_{f,T}\right) \ominus \Theta_{f,T}(\mathbb{H}^2(f) \otimes \mathcal{D}_{f,T^*}), \qquad \mathbf{T}_i^* x = (M_{Z_i}^* \otimes I_{\mathcal{D}_{f,T}})x, \qquad x \in \mathbf{H}.$$

*Proof.* If $T := (T_1, \ldots, T_n)$ is in $\mathbb{B}_f(\mathcal{H})^{cnc}$, then the $n$-tuple $f(T) := (f_1(T), \ldots, f_n(T))$ is a c.n.c row contraction. According to [11], $f(T)$ is unitarily equivalent to a row contraction $\mathbf{A} := (\mathbf{A}_1, \ldots, \mathbf{A}_n)$ on the Hilbert space

$$\widetilde{\mathcal{H}} := [(F^2(H_n) \otimes \mathcal{D}_{f(T)}) \oplus \overline{\Delta_{\Theta_{f(T)}}(F^2(H_n) \otimes \mathcal{D}_{f(T)^*})}] \ominus \{\Theta_{f(T)} z \oplus \Delta_{\Theta_{f(T)}} z : \quad z \in F^2(H_n) \otimes \mathcal{D}_{f(T)^*}\},$$

where $\Theta_{f(T)}$ is the characteristic function of the row contraction $f(T) := (f_1(T), \ldots, f_n(T))$, the defect operator $\Delta_{\Theta_{f(T)}} := (I - \Theta_{f(T)}^* \Theta_{f(T)})^{1/2}$, and the operator $\mathbf{A}_i$ is defined on $\widetilde{\mathcal{H}}$ by setting

$$\mathbf{A}_i^*[\omega \oplus \Delta_{\Theta_{f(T)}} z] := (S_i^* \otimes I_{\mathcal{D}_{f(T)}})\omega \oplus C_i^*(\Delta_{\Theta_{f(T)}} z), \qquad i = 1, \ldots, n, \tag{2.1}$$

for $\omega \in F^2(H_n) \otimes \mathcal{D}_{f(T)}$, $z \in F^2(H_n) \otimes \mathcal{D}_{f(T)^*}$, where $C_i$ is defined on $\overline{\Delta_{\Theta_{f(T)}}(F^2(H_n) \otimes \mathcal{D}_{f(T)^*})}$ by

$$C_i(\Delta_{\Theta_{f(T)}} z) := \Delta_{\Theta_{f(T)}}(S_i \otimes I_{\mathcal{D}_{f(T)^*}})z, \qquad i = 1, \ldots, n,$$

and $S_1, \ldots, S_n$ are the left creation operators on the full Fock space $F^2(H_n)$. Since $f(T)$ and $\mathbf{A}$ are completely non-coisometric row contractions and $g$, the inverse of $f$ with respect to the composition, is a bounded free holomorphic function on the unit ball $[B(\mathcal{H})^n]_1$, then, using the functional calculus for c.n.c. row contractions (see [13]), it makes sense to talk about $g(\mathbf{A}) := (g_1(\mathbf{A}), \ldots, g_n(\mathbf{A}))$ and $g(f(T)) := (g_1(f(T)), \ldots, g_n(f(T)))$. Consequently, since $g(f(T)) = T$ and $f(T)$ is unitarily equivalent to $\mathbf{A}$, we deduce that $T = (T_1, \ldots, T_n)$ is unitarily equivalent to $\mathbb{T} := (\mathbb{T}_1, \ldots, \mathbb{T}_n)$, where $\mathbb{T}_i = g_i(\mathbf{A})$. Since $f \in \mathcal{M}^b$, we use Proposition 2.2 to conclude that $\mathbb{T} \in \mathbb{B}_f^{cnc}(\widetilde{\mathcal{H}})$.

Consider the canonical unitary operator $U : \mathbb{H}^2(f) \to F^2(H_n)$ defined by $U f_\alpha = e_\alpha$, $\alpha \in \mathbb{F}_n^+$ and note that

$$\Theta_{f,T} = (U^* \otimes I_{\mathcal{D}_{f,T}})\Theta_{f(T)}(U \otimes I_{\mathcal{D}_{f,T^*}}), \tag{2.2}$$

where $\Theta_{f(T)}$ is the characteristic function of the row contraction $f(T) = [f_1(T), \ldots, f_n(T)]$. Hence, we deduce that

$$\Delta_{\Theta_{f,T}} = (U^* \otimes I_{\mathcal{D}_{f,T^*}})\Delta_{\Theta_{f(T)}}(U \otimes I_{\mathcal{D}_{f,T^*}}). \tag{2.3}$$

Define the subspaces

$$\mathbf{G} := \{\Theta_{f,T} x \oplus \Delta_{\Theta_{f,T}} x : \quad x \in \mathbb{H}^2(f) \otimes \mathcal{D}_{f,T^*}\}$$

and

$$\widetilde{G} := \{\Theta_{f(T)} z \oplus \Delta_{\Theta_{f(T)}} z : \quad z \in F^2(H_n) \otimes \mathcal{D}_{f(T)^*}\},$$

and the unitary operator $\Gamma$ acting from the Hilbert space $(\mathbb{H}^2(f) \otimes \mathcal{D}_{f,T}) \oplus \overline{\Delta_{\Theta_{f,T}}(\mathbb{H}^2(f) \otimes \mathcal{D}_{f,T^*})}$ to $(F^2(H_n) \otimes \mathcal{D}_{f(T)}) \oplus \overline{\Delta_{\Theta_{f(T)}}(F^2(H_n) \otimes \mathcal{D}_{f(T)^*})}$ and defined by

$$\Gamma := (U \otimes I_{\mathcal{D}_{f,T}}) \oplus (U \otimes I_{\mathcal{D}_{f,T^*}}).$$

Since $\mathcal{D}_{f,T} = \mathcal{D}_{f(T)}$ and $\mathcal{D}_{f,T^*} = \mathcal{D}_{f(T)^*}$, it is easy to see that that $\Gamma(\mathbf{G}) = \widetilde{\mathcal{G}}$ and $\Gamma(\mathbf{H}) = \widetilde{\mathcal{H}}$. Therefore, $\Gamma|_{\mathbf{H}} : \mathbf{H} \to \widetilde{\mathcal{H}}$ is a unitary operator.

We introduce the operators $\mathbf{B}_i := (\Gamma|_{\mathbf{H}})^{-1}\mathbb{T}_i(\Gamma|_{\mathbf{H}})$, $i = 1, \ldots, n$. Since the $n$-tuple $(\mathbb{T}_1, \ldots, \mathbb{T}_n)$ is in $\mathbb{B}_f^{cnc}(\widetilde{\mathcal{H}})$, we deduce that $\mathbf{B} := (\mathbf{B}_1, \ldots, \mathbf{B}_n)$ is in $\mathbb{B}_f^{cnc}(\mathbf{H})$.

Now, we show that the operators $\mathbf{T}_i$, $i = 1, \ldots, n$, defined in the theorem are well-defined and bounded on the Hilbert space $\mathbf{H}$. Note that since $\Theta_{f,T}$ is a multi-analytic operator with respect to $M_{Z_1}, \ldots, M_{Z_n}$, we have

$$\begin{aligned}
\left\|D_i(\Delta_{\Theta_{f,T}} y)\right\| &= \left\|\Delta_{\Theta_{f,T}}(M_{Z_i} \otimes I_{\mathcal{D}_{f,T^*}})y\right\| \\
&= \left\langle M_{Z_i}^* M_{Z_i} \otimes I_{\mathcal{D}_{f,T^*}} - \Theta_{f,T}^* M_{Z_i}^* M_{Z_i} \Theta_{f,T})y, y\right\rangle \\
&= \|Z_i\|_{\mathbb{H}^2(f)}^2 \left\langle (I - \Theta_{f,T}^* \Theta_{f,T})y, y\right\rangle = \|Z_i\|_{\mathbb{H}^2(f)}^2 \left\|\Delta_{\Theta_{f,T}} y\right\|^2
\end{aligned}$$

for any $y \in \mathbb{H}^2(f)\otimes \mathcal{D}_{f,T^*}$. Consequently, $D_i$ extends to a unique bounded operator on the Hilbert space $\overline{\Delta_{\Theta_{f,T}}(\mathbb{H}^2(f)\otimes \mathcal{D}_{f,T^*})}$. Note also that due to the fact that the subspace $\mathbf{G}$ is invariant under $(M_{Z_i}\otimes I_{\mathcal{D}_{f,T^*}})\oplus D_i$, we have $\left[(M_{Z_i}^*\otimes I_{\mathcal{D}_{f,T^*}})\oplus D_i^*\right](\mathbf{H})\subset \mathbf{H}$, which proves our assertion.

Our next step is to show that $\mathbf{B}_i=\mathbf{T}_i$ for $i=1,\dots,n$. First, note that due to relation (2.1) and the functional calculus for c.n.c. row contractions, we have

$$\begin{aligned}
&\left(\omega\oplus \Delta_{\Theta_{f(T)}}z, \mathbb{T}_i(\omega'\oplus \Delta_{\Theta_{f(T)}}z')\right)\\
&\quad=\left(\omega\oplus \Delta_{\Theta_{f(T)}}z, g_i(\mathbf{A}_1,\dots,\mathbf{A}_n)(\omega'\oplus \Delta_{\Theta_{f(T)}}z')\right)\\
&\quad=\lim_{r\to 1}\left\langle \omega\oplus \Delta_{\Theta_{f(T)}}z,\ g_i\left(r[(S_1\otimes I_{\mathcal{D}_{f(T)}})\oplus C_1],\dots,r[(S_n\otimes I_{\mathcal{D}_{f(T)}})\oplus C_n)]\right)(\omega'\oplus \Delta_{\Theta_{f(T)}}z')\right\rangle\\
&\quad=\lim_{r\to 1}\left\langle \omega\oplus \Delta_{\Theta_{f(T)}}z,\ \left[(g_i(rS_1,\dots,rS_n)\otimes I_{\mathcal{D}_{f(T)}})\oplus g_i(rC_1,\dots,rC_n)\right](\omega'\oplus \Delta_{\Theta_{f(T)}}z')\right\rangle\\
&\quad=\lim_{r\to 1}\left\langle \omega\oplus \Delta_{\Theta_{f(T)}}z,\ [g_i(rS_1,\dots,rS_n)\otimes I_{\mathcal{D}_{f(T)}}]\omega'\oplus [\Delta_{\Theta_{f(T)}}(g_i(rS_1,\dots,rS_n)\otimes I_{\mathcal{D}_{f(T)^*}})]z'\right\rangle
\end{aligned}$$

for any $\omega,\omega'\in F^2(H_n)\otimes \mathcal{D}_{f(T)}$ and $z,z'\in F^2(H_n)\otimes \mathcal{D}_{f(T)^*}$.

Now, using relation (2.3), for any $x,x'\in \mathbb{H}^2(f)\otimes \mathcal{D}_{f,T}$ and $y,y'\in \mathbb{H}^2(f)\otimes \mathcal{D}_{f,T^*}$, we have

$$\begin{aligned}
&\left\langle x\oplus \Delta_{\Theta_{f,T}}y, \mathbf{B}_i(x'\oplus \Delta_{\Theta_{f,T}}y')\right\rangle\\
&\quad=\left\langle x\oplus \Delta_{\Theta_{f,T}}y, (\Gamma|_{\mathbf{H}})^{-1}\mathbb{T}_i(\Gamma|_{\mathbf{H}})(x'\oplus \Delta_{\Theta_{f,T}}y')\right\rangle\\
&\quad=\left\langle [(U\otimes I_{\mathcal{D}_{f,T}})x\oplus \Delta_{\Theta_{f,T}}(U\otimes I_{\mathcal{D}_{f,T}})y], \mathbb{T}_i[(U\otimes I_{\mathcal{D}_{f,T}})x'\oplus \Delta_{\Theta_{f,T}}(U\otimes I_{\mathcal{D}_{f,T}})y']\right\rangle.
\end{aligned}$$

Setting $\omega=(U\otimes I_{\mathcal{D}_{f,T}})x$, $z=(U\otimes I_{\mathcal{D}_{f,T}})y$, $\omega'=(U\otimes I_{\mathcal{D}_{f,T}})x'$, $z'=(U\otimes I_{\mathcal{D}_{f,T}})y'$, and combining the results above, we obtain

$$\begin{aligned}
&\left\langle x\oplus \Delta_{\Theta_{f,T}}y, \mathbf{B}_i(x'\oplus \Delta_{\Theta_{f,T}}y')\right\rangle\\
&\quad=\lim_{r\to 1}\left\langle [(U\otimes I_{\mathcal{D}_{f,T}})x\oplus \Delta_{\Theta_{f,T}}(U\otimes I_{\mathcal{D}_{f,T}})y],\right.\\
&\qquad\left.[g_i(rS_1,\dots,rS_n)\otimes I_{\mathcal{D}_{f(T)}}](U\otimes I_{\mathcal{D}_{f,T}})x'\oplus [\Delta_{\Theta_{f(T)}}(g_i(rS_1,\dots,rS_n)\otimes I_{\mathcal{D}_{f(T)^*}})](U^*\otimes I_{\mathcal{D}_{f,T}})y'\right\rangle\\
&\quad=\lim_{r\to 1}\left\langle x\oplus \Delta_{\Theta_{f,T}}y, [(U^*g_i(rS_1,\dots,rS_n)U)\otimes I_{\mathcal{D}_{f,T}}]x'\oplus [\Delta_{\Theta_{f(T)}}(U^*g_i(rS_1,\dots,rS_n)U\otimes I_{\mathcal{D}_{f(T)^*}})]y'\right\rangle\\
&\quad=\lim_{r\to 1}\left\langle x\oplus \Delta_{\Theta_{f,T}}y, [g_i(rM_{f_1},\dots,rM_{f_n})\otimes I_{\mathcal{D}_{f,T}}]x'\oplus [\Delta_{\Theta_{f(T)}}(g_i(rM_{f_1},\dots,rM_{f_n})\otimes I_{\mathcal{D}_{f(T)^*}})]y'.\right\rangle
\end{aligned}$$

On the other hand, since $f$ has the model property, we have

$$M_{Z_i}=g_i(M_{f_1},\dots,M_{f_n})=\text{SOT-}\lim_{r\to 1}g_i(rM_{f_1},\dots,rM_{f_n}).$$

Now, we can deduce that

$$\begin{aligned}
&\left\langle x\oplus \Delta_{\Theta_{f,T}}y, \mathbf{B}_i(x'\oplus \Delta_{\Theta_{f,T}}y')\right\rangle\\
&\quad=\left\langle x\oplus \Delta_{\Theta_{f,T}}y, (M_{Z_i}\otimes I_{\mathcal{D}_{f,T}})x'\oplus \Delta_{\Theta_{f(T)}}(M_{Z_i}\otimes I_{\mathcal{D}_{f,T}})y'\right\rangle\\
&\quad=\left\langle x\oplus \Delta_{\Theta_{f,T}}y, \mathbf{T}_i(x\oplus \Delta_{\Theta_{f,T}}y')\right\rangle
\end{aligned}$$

for any $x,x'\in \mathbb{H}^2(f)\otimes \mathcal{D}_{f,T}$ and $y,y'\in \mathbb{H}^2(f)\otimes \mathcal{D}_{f,T^*}$. Hence, we obtain $\mathbf{B}_i=\mathbf{T}_i$ for any $i=1,\dots,n$, which completes the first part of the proof.

Since the $n$-tuples of operators $T:=(T_1,\dots,T_n)$ and $\mathbb{T}:=(\mathbb{T}_1,\dots,\mathbb{T}_n)=(g_1(\mathbf{A}),\dots,g_n(\mathbf{A}))$ are unitarily equivalent, we deduce that $T\in \mathbb{B}_f^{pure}(\mathcal{H})$ if and only if $\mathbb{T}\in \mathbb{B}_f^{pure}(\widetilde{\mathcal{H}})$. On the other hand, due to Proposition 2.2, $\mathbb{T}\in \mathbb{B}_f^{pure}(\widetilde{\mathcal{H}})$ if and only if $\mathbf{A}\in [B(\mathcal{H})^n]_1^{pure}$. Since the row contraction $f(T):=(f_1(T),\dots,f_n(T))$ is unitarily equivalent to $\mathbf{A}$, Theorem 4.1 from [11] shows that $\mathbf{A}$ is pure if and only if the characteristic function $\Theta_{f(T)}$ is an isometry which, due to relation (2.2), is equivalent to $\Theta_{f,T}$ being an isometry. This completes the proof. $\square$

Let $\Phi:\mathbb{H}^2(f)\otimes \mathcal{K}_1\to \mathbb{H}^2(f)\otimes \mathcal{K}_2$ and $\Phi':\mathbb{H}^2(f)\otimes \mathcal{K}_1'\to \mathbb{H}^2(f)\otimes \mathcal{K}_2'$ be two multi-analytic operators with respect to $M_{Z_1},\dots,M_{Z_n}$. We say that $\Phi$ and $\Phi'$ coincide if there are two unitary operators $\tau_j\in B(\mathcal{K}_j,\mathcal{K}_j')$, $j=1,2$, such that

$$\Phi'(I_{\mathbb{H}^2(f)}\otimes \tau_1)=(I_{\mathbb{H}^2(f)}\otimes \tau_2)\Phi.$$

The next result shows that the characteristic function is a complete unitary invariant for the $n$-tuples of operators in the c.n.c. part of the noncommutative domain $\mathbb{B}_f(\mathcal{H})$.

**Theorem 2.4.** *Let $f=(f_1,\dots,f_n)$ be an $n$-tuple of formal power series in the class $\mathcal{M}^b$ and let $T:=(T_1,\dots,T_n)\in \mathbb{B}_f^{cnc}(\mathcal{H})$ and $T':=(T_1',\dots,T_n')\in \mathbb{B}_f^{cnc}(\mathcal{H}')$. Then $T$ and $T'$ are unitarily equivalent if and only if their characteristic functions $\Theta_{f,T}$ and $\Theta_{f,T'}$ coincide.*

*Proof.* Assume that $T$ and $T'$ are unitarily equivalent and let $W:\mathcal{H}\to\mathcal{H}'$ be a unitary operator such that $T_i=W^*T_i'W$ for any $i=1,\dots,n$. Since $T\in\mathcal{C}_f^{SOT}(\mathcal{H})$ or $T\in\mathcal{C}_f^{rad}(\mathcal{H})$ and similar relations hold for $T'$, it is easy to see that

$$W\Delta_{f,T}=\Delta_{f,T'}W \quad \text{and} \quad (\oplus_{i=1}^n W)\Delta_{f,T^*}=\Delta_{f,T'^*}(\oplus_{i=1}^n W).$$

Define the unitary operators $\tau$ and $\tau'$ by setting

$$\tau:=W|_{\mathcal{D}_{f,T}}:\mathcal{D}_{f,T}\to\mathcal{D}_{f,T'} \quad \text{and} \quad \tau':=(\oplus_{i=1}^n W)|_{\mathcal{D}_{f,T^*}}:\mathcal{D}_{f,T^*}\to\mathcal{D}_{f,T'^*}.$$

Using the definition of the characteristic function, we deduce that that

$$(I_{\mathbb{H}^2(f)}\otimes\tau)\Theta_{f,T}=\Theta_{f,T'}(I_{\mathbb{H}^2(f)}\otimes\tau').$$

Conversely, assume that the characteristic functions of $T$ and $T'$ coincide. Then there exist unitary operators $\tau:\mathcal{D}_{f,T}\to\mathcal{D}_{f,T'}$ and $\tau_*:\mathcal{D}_{f,T^*}\to\mathcal{D}_{f,T'^*}$ such that

$$(I_{\mathbb{H}^2(f)}\otimes\tau)\Theta_{f,T}=\Theta_{f,T'}(I_{\mathbb{H}^2(f)}\otimes\tau_*). \tag{2.4}$$

Hence, we obtain

$$\Delta_{\Theta_{f,T}}=\left(I_{\mathbb{H}^2(f)}\otimes\tau_*\right)^*\Delta_{\Theta_{f,T'}}\left(I_{\mathbb{H}^2(f)}\otimes\tau_*\right) \tag{2.5}$$

and

$$\left(I_{\mathbb{H}^2(f)}\otimes\tau_*\right)\overline{\Delta_{f,T}(\mathbb{H}^2(f)\otimes\mathcal{D}_{f,T^*})}=\overline{\Delta_{f,T'}(\mathbb{H}^2(f)\otimes\mathcal{D}_{f,T'^*})}.$$

Consider the Hilbert spaces

$$\mathbf{K}_{f,T}:=[(\mathbb{H}^2(f)\otimes\mathcal{D}_{f,T})\oplus\overline{\Delta_{\Theta_{f,T}}(\mathbb{H}^2(f)\otimes\mathcal{D}_{f,T^*})}],$$

$$\mathbf{G}_{f,T}:=\{\Theta_{f,T}x\oplus\Delta_{\Theta_{f,T}}x:\quad x\in\mathbb{H}^2(f)\otimes\mathcal{D}_{f,T^*}\},$$

and $\mathbf{H}_{f,T}:=\mathbf{K}_{f,T}\ominus\mathbf{G}_{f,T}$. We define the unitary operator $\Gamma:\mathbf{K}_{f,T}\to\mathbf{K}_{f,T'}$ by setting

$$\Gamma:=(I_{\mathbb{H}^2(f)}\otimes\tau)\oplus(I_{\mathbb{H}^2(f)}\otimes\tau_*).$$

Due to relations (2.4) and (2.5), we have $\Gamma(\mathbf{G}_{f,T})=\mathbf{G}_{f,T'}$ and $\Gamma(\mathbf{H}_{f,T})=\mathbf{H}_{f,T'}$. Therefore, the operator $\Gamma|_{\mathbf{H}_{f,T}}:\mathbf{H}_{f,T}\to\mathbf{H}_{f,T'}$ is unitary. Now, let $\mathbf{T}:=[\mathbf{T}_1,\dots\mathbf{T}_n]$ and $\mathbf{T}':=[\mathbf{T}_1',\dots\mathbf{T}_n']$ be the models provided by Theorem 2.3 for the $n$-tuples $T$ and $T'$, respectively.

We recall that the operator $D_i$ is defined by $D_i(\Delta_{\Theta_{f,T}}y):=\Delta_{\Theta_{f,T}}(M_{Z_i}\otimes I_{\mathcal{D}_{f,T^*}})y$ for all $y\in\mathbb{H}^2(f)\otimes\mathcal{D}_{f,T^*}$. Using relation (2.5), we deduce that

$$\begin{aligned}
D_i'((I\otimes\tau_*)\Delta_{\Theta_{f,T}}y)&=D_i'(\Delta_{\Theta_{f,T'}}(I\otimes\tau_*)y)\\
&=\Delta_{\Theta_{f,T'}}(M_{Z_i}\otimes I_{\mathcal{D}_{f,T'^*}})(I\otimes\tau_*)y)\\
&=(I\otimes\tau_*)\Delta_{\Theta_{f,T}}(M_{Z_i}\otimes I_{\mathcal{D}_{f,T^*}})y\\
&=(I\otimes\tau_*)D_i(\Delta_{\Theta_{f,T}}y).
\end{aligned}$$

Consequently, we obtain that $D_i'^*(I\otimes\tau_*)=(I\otimes\tau_*)D_i^*$. On the other hand, we have

$$(M_{Z_i}^*\otimes I_{\mathcal{D}_{f,T'}})(I_{\mathbb{H}^2(f)}\otimes\tau)=(I_{\mathbb{H}^2(f)}\otimes\tau)(M_{Z_i}^*\otimes I_{\mathcal{D}_{f,T}}).$$

Combining these relations and using the definition of the unitary operator $\Gamma$, we deduce that

$$\begin{aligned}
\mathbf{T}_i'^*\Gamma(x+\Delta_{\Theta_{f,T}}y) &= \mathbf{T}_i'^*\left((I_{\mathbb{H}^2(f)}\otimes\tau)x\oplus(I_{\mathbb{H}^2(f)}\otimes\tau_*)\Delta_{\Theta_{f,T}}y\right)\\
&= \mathbf{T}_i'^*\left((I_{\mathbb{H}^2(f)}\otimes\tau)x\oplus\Delta_{\Theta_{f,T'}}(I_{\mathbb{H}^2(f)}\otimes\tau_*)y\right)\\
&= (M_{Z_i}^*\otimes I_{\mathcal{D}_{f,T'}})(I_{\mathbb{H}^2(f)}\otimes\tau)x\oplus D_i'^*\left(\Delta_{\Theta_{f,T'}}(I_{\mathbb{H}^2(f)}\otimes\tau_*)y\right)\\
&= (M_{Z_i}^*\otimes I_{\mathcal{D}_{f,T'}})(I_{\mathbb{H}^2(f)}\otimes\tau)x\oplus D_i'^*\left((I_{\mathbb{H}^2(f)}\otimes\tau_*)\Delta_{\Theta_{f,T}}y\right)\\
&= (I_{\mathbb{H}^2(f)}\otimes\tau)(M_{Z_i}^*\otimes I_{\mathcal{D}_{f,T}})x\oplus(I_{\mathbb{H}^2(f)}\otimes\tau_*)D_i'^*\left(\Delta_{\Theta_{f,T}}y\right)\\
&= \Gamma\mathbf{T}_i^*(x\oplus\Delta_{\Theta_{f,T}}y)
\end{aligned}$$

for any $x\oplus\Delta_{\Theta_{f,T}}y\in\mathbf{H}_{f,T}$ and $i=1,\ldots,n$. Consequently, we obtain $\mathbf{T}_i'^*(\Gamma|_{\mathbf{H}_{f,T}})=(\Gamma|_{\mathbf{H}_{f,T}})\mathbf{T}_i^*$ for $i=1,\ldots,n$. Now, using Theorem 2.3, we conclude that $T$ and $T'$ are unitarily equivalent. The proof is complete. $\square$

In what follows we prove that any contractive multi-analytic operator $\Theta:\mathbb{H}^2(f)\otimes\mathcal{E}_*\to\mathbb{H}^2(f)\otimes\mathcal{E}$ ($\mathcal{E},\mathcal{E}_*$ are Hilbert spaces) with respect to $M_{Z_1},\ldots,M_{Z_n}$ generates a c.n.c. $n$-tuple of operators $\mathbf{T}:=(\mathbf{T}_1,\ldots,\mathbf{T}_n)\in\mathbb{B}_f(\mathbf{H})$. We mention that $\Theta$ is called purely contractive if $\|P_{\mathcal{E}}\Theta(1\otimes x)\|<\|x\|$ for any $x\in\mathcal{E}_*$.

**Theorem 2.5.** *Let $f=(f_1,\ldots,f_n)$ be an $n$-tuple of formal power series in the class $\mathcal{M}^b$ and let $(M_{Z_1},\ldots,M_{Z_n})$ be the universal model associated with the noncommutative domain $\mathbb{B}_f$. Let $\Theta:\mathbb{H}^2(f)\otimes\mathcal{E}_*\to\mathbb{H}^2(f)\otimes\mathcal{E}$ be a contractive multi-analytic operator and set $\Delta_\Theta:=(I-\Theta^*\Theta)^{1/2}$. Then the $n$-tuple $\mathbf{T}:=(\mathbf{T}_1,\ldots,\mathbf{T}_n)$ defined on the Hilbert space*

$$\mathbf{H}:=[(\mathbb{H}^2(f)\otimes\mathcal{E})\oplus\overline{\Delta_\Theta(\mathbb{H}^2(f)\otimes\mathcal{E}_*)}]\ominus\{\Theta y\oplus\Delta_\Theta y:\quad y\in\mathbb{H}^2(f)\otimes\mathcal{E}_*\}$$

*by*

$$\mathbf{T}_i^*(x\oplus\Delta_\Theta y):=(M_{Z_i}^*\otimes I_{\mathcal{E}_*})x\oplus D_i^*(\Delta_\Theta y),\qquad i=1,\ldots,n,$$

*where each operator $D_i$ is defined by*

$$D_i(\Delta_\Theta y):=\Delta_\Theta(M_{Z_i}\otimes I_{\mathcal{E}})y,\quad y\in\mathbb{H}^2(f)\otimes\mathcal{E}_*,$$

*is in $\mathbb{B}_f^{cnc}(\mathbf{H})$.*

*Moreover, if $\Theta$ is purely contractive and*

$$\overline{\Delta_\Theta(\mathbb{H}^2(f)\otimes\mathcal{E}_*)}=\overline{\Delta_\Theta(\vee_{i=1}^n M_{f_i}(\mathbb{H}^2(f)\otimes\mathcal{E}_*))},$$

*then $\Theta$ coincides with the characteristic function of the $n$-tuple $\mathbf{T}:=(\mathbf{T}_1,\ldots,\mathbf{T}_n)$.*

*Proof.* Since $f=(f_1,\ldots,f_n)$ has the model property, we have $M_{f_j}=f_j(M_{Z_1},\ldots,M_{Z_n})$ where the $n$-tuple $(M_{Z_1},\ldots,M_{Z_n})$ is either in the convergence set $\mathcal{C}_f^{SOT}(\mathbb{H}^2(f))$ or $\mathcal{C}_f^{rad}(\mathbb{H}^2(f))$, and $g_j(M_{f_1},\ldots,M_{f_n})=M_{Z_j}$ where $(M_{f_1},\ldots,M_{f_n})$ is in the convergence set $\mathcal{C}_g^{rad}(\mathbb{H}^2(f))$. Consequently, we have $\Theta(M_{Z_i}\otimes I_{\mathcal{E}_*})=(M_{Z_i}\otimes I_{\mathcal{E}})$, $i=1,\ldots,n$, if and only if $\Theta(M_{f_i}\otimes I_{\mathcal{E}_*})=(M_{f_i}\otimes I_{\mathcal{E}})$, $i=1,\ldots,n$. Setting $\Psi:=(U\otimes I_{\mathcal{E}})\Theta(U^*\otimes I_{\mathcal{E}_*})$, where the canonical unitary operator $U:\mathbb{H}^2(f)\to F^2(H_n)$ is defined by $Uf_\alpha=e_\alpha$, $\alpha\in\mathbb{F}_n^+$, we deduce that $\Psi$ is a multi-analytic operator on the Fock space $F^2(H_n)$, i.e., $\Psi(S_i\otimes I_{\mathcal{E}_*})=(S_i\otimes I_{\mathcal{E}})$ for $i=1,\ldots,n$.

Define the $n$-tuple $\mathbf{A}:=(\mathbf{A}_1,\ldots,\mathbf{A}_n)$ on the Hilbert space

$$\widetilde{\mathcal{H}}:=[(F^2(H_n)\otimes\mathcal{E})\oplus\overline{\Delta_\Psi(F^2(H_n)\otimes\mathcal{E}_*)}]\ominus\{\Psi z\oplus\Delta_\Psi z:\quad z\in F^2(H_n)\otimes\mathcal{E}_*\},$$

where the defect operator $\Delta_\Psi:=(I-\Psi^*\Psi)^{1/2}$ and the operator $\mathbf{A}_i$, $i=1,\ldots,n$, is defined on $\widetilde{\mathcal{H}}$ by setting

$$\mathbf{A}_i^*[\omega\oplus\Delta_\Psi z]:=(S_i^*\otimes I_{\mathcal{E}})\omega\oplus C_i^*(\Delta_\Psi z),\qquad i=1,\ldots,n,$$

for $\omega\in F^2(H_n)\otimes\mathcal{E}$, $z\in F^2(H_n)\otimes\mathcal{E}_*$, where $C_i$ is defined on $\overline{\Delta_\Psi(F^2(H_n)\otimes\mathcal{E}_*)}$ by

$$C_i(\Delta_\Psi z):=\Delta_\Psi(S_i\otimes I_{\mathcal{E}_*})z,\qquad i=1,\ldots,n,$$

and $S_1,\ldots,S_n$ are the left creation operators on the full Fock space $F^2(H_n)$.

Consider the Hilbert spaces

$$\widetilde{\mathcal{K}} := (F^2(H_n)\otimes \mathcal{E}) \oplus \overline{\Delta_\Psi(F^2(H_n)\otimes \mathcal{E}_*)}$$

and

$$\widetilde{\mathcal{G}} := \{\Psi z \oplus \Delta_\Psi z : \quad z\in F^2(H_n)\otimes \mathcal{E}_*\}.$$

Since $\Psi$ is a multi-analytic operator on the Fock space $F^2(H_n)$, it is easy to see the $[C_1,\ldots,C_n]$ is a row isometry and $\widetilde{\mathcal{G}}$ is invariant under each operator $\mathbf{W}_i := S_i\oplus C_i$, $i=1,\ldots,n$, acting on $\widetilde{\mathcal{K}}$. Therefore, $\mathbf{A}_i^* = \mathbf{W}_i^*|_{\widetilde{\mathcal{H}}}$, $i=1,\ldots,n$. Note also that $\mathbf{A}=[\mathbf{A}_1,\ldots,\mathbf{A}_n]$ is a c.n.c. row contraction. Indeed, let $\omega\oplus\Delta_\Psi z\in\widetilde{\mathcal{H}}$ be such that $\sum_{|\alpha|=k}\|\mathbf{A}_\alpha^*(\omega\oplus\Delta_\Psi z)\|^2 = \|\omega\oplus\Delta_\Psi z\|^2$ for any $k\in\mathbb{N}$. Taking into account that $\lim_{k\to\infty}\sum_{|\alpha|=k}\|S_\alpha^*\omega\|^2=0$ and $\sum_{|\alpha|=k}\|C_\alpha^*\Delta_\Psi z\|^2\leq \|\Delta_\Psi z\|^2$, we deduce that $\omega=0$. On the other hand, since $0\oplus\Delta_\Psi z\in\widetilde{\mathcal{H}}$, we must have $\langle 0\oplus\Delta_\Psi z, \Psi u\oplus\Delta_\Psi u\rangle = 0$ for any $u\in F^2(H_n)\otimes\mathcal{E}_*$, which implies $\Delta_\Psi z=0$ and proves our assertion.

Since $\mathbf{A}$ is a completely non-coisometric row contractions and $g$, the inverse of $f$ with respect to the composition, is a bounded free holomorphic function on the unit ball $[B(\mathcal{H})^n]_1$, it makes sense to talk about $g(\mathbf{A}) := (g_1(\mathbf{A}),\ldots,g_n(\mathbf{A}))$ using the functional calculus for c.n.c. row contractions (see [13]). Since $f\in\mathcal{M}^b$, setting $\mathbb{T} := (\mathbb{T}_1,\ldots,\mathbb{T}_n)$, where $\mathbb{T}_i = g_i(\mathbf{A})$, and using Proposition 2.2, we deduce that $\mathbb{T}\in\mathbb{B}_f^{cnc}(\widetilde{\mathcal{H}})$. Consider the unitary operator $\Gamma$ acting from the Hilbert space $(\mathbb{H}^2(f)\otimes\mathcal{E})\oplus\overline{\Delta_\Theta(\mathbb{H}^2(f)\otimes\mathcal{E}_*)}$ to $(F^2(H_n)\otimes\mathcal{E})\oplus\overline{\Delta_\Psi(F^2(H_n)\otimes\mathcal{E}_*)}$ and defined by

$$\Gamma := (U\otimes I_{\mathcal{E}})\oplus(U\otimes I_{\mathcal{E}_*}).$$

As in the proof of Theorem 2.3, one can show that $\mathbf{T}_i$ is a bounded operator on $\mathbf{H}$ and $\mathbf{T}_i = (\Gamma|_{\mathbf{H}})^{-1}\mathbb{T}_i(\Gamma|_{\mathbf{H}})$, $i=1,\ldots,n$. Consequently, we have $\mathbf{T}\in\mathbb{B}_f^{cnc}(\mathbf{H})$, which proves the first part of the theorem.

To prove the second part of the theorem, we assume that $\Theta$ is purely contractive, i.e., $\|P_{\mathcal{E}}\Theta(1\otimes x)\| < \|x\|$ for any $x\in\mathcal{E}_*$, and

$$\overline{\Delta_\Theta(\mathbb{H}^2(f)\otimes\mathcal{E}_*)} = \overline{\Delta_\Theta(\vee_{i=1}^n M_{f_i}(\mathbb{H}^2(f)\otimes\mathcal{E}_*))}.$$

These conditions imply that $\Psi$ is purely contractive and

$$\overline{\Delta_\Psi(F^2(H_n)\otimes\mathcal{E}_*)} = \overline{\Delta_\Psi[(F^2(H_n)\otimes\mathcal{E}_*)\ominus\mathcal{E}_*]}.$$

According to Theorem 4.1 from [11], the multi-analytic operator $\Psi$ coincides with the characteristic function $\Theta_{\mathbf{A}}$ of the row contraction $\mathbf{A}=[\mathbf{A}_1,\ldots,\mathbf{A}_n]$.

Note that the characteristic function of $\mathbb{T}=g(\mathbf{A})\in\mathbb{B}_{\mathbf{f}}^{\mathbf{cnc}}(\widetilde{\mathcal{H}})$ is

$$\Theta_{f,\mathbb{T}} = (U^*\otimes I)\Theta_{f(\mathbb{T})}(U\otimes I) = (U^*\otimes I)\Theta_{\mathbf{A}}(U\otimes I).$$

Since $\mathbf{T}_i = (\Gamma|_{\mathbf{H}})^{-1}\mathbb{T}_i(\Gamma|_{\mathbf{H}})$, $i=1,\ldots,n$, we also have that the characteristic functions $\Theta_{f,\mathbf{T}}$ and $\Theta_{f,\mathbb{T}}$ coincide. Combining these results with the fact that $(U^*\otimes I_{\mathcal{E}_*})\Psi(U\otimes I_{\mathcal{E}}) = \Theta$, we conclude that $\Theta$ coincides with $\Theta_{f,\mathbf{T}}$. The proof is complete. □

## 3. Dilation theory on noncommutative domains

In this section, we study the $*$-representations of the $C^*$-algebra $C^*(M_{Z_1},\ldots,M_{Z_n})$ and obtain a Wold type decomposition for the nondegenerate $*$-representations. Under natural conditions on the $n$-tuple $f=(f_1,\ldots,f_n)$ of formal power series, we show that any $n$-tuple $T=(T_1,\ldots,T_n)$ of operators is in the noncommutative domain $\mathbb{B}_f(\mathcal{H})$, has a minimal dilation which is unique up to an isomorphism. We also provide a commutant lifting theorem for $\mathbb{B}_f(\mathcal{H})$.

**Proposition 3.1.** *Let $f=(f_1,\ldots,f_n)$ be an $n$-tuple of formal power series in the class $\mathcal{M}^{||}$ and let $(M_{Z_1},\ldots,M_{Z_n})$ be the universal model associated with $\mathbb{B}_f$. Then all compact operators in $B(\mathbb{H}^2(f))$ are contained in $C^*(M_{Z_1},\ldots,M_{Z_n})$.*

*Proof.* Since $f \in \mathcal{M}^{||}$, the universal $n$-tuple $(M_{Z_1}, \ldots, M_{Z_n})$ is in the set of norm-convergence (or radial norm-convergence) for $f$ and, consequently, the operator $f_i(M_Z)$ is in $\overline{\text{span}}\{M_{Z_\alpha} M_{Z_\beta}^* : \alpha, \beta \in \mathbb{F}_n^+\}$. Taking into account that $f = (f_1, \ldots, f_n)$ is an $n$-tuple of formal power series with the model property, we have $M_{f_i} = f_i(M_{Z_1}, \ldots, M_{Z_n})$. On the other hand, the orthogonal projection of $\mathbb{H}^2(f)$ onto the constant power series satisfies the equation $P_{\mathbb{C}} = I - \sum_{i=1}^n f_i(M_Z) f_i(M_Z)^*$. Therefore, $P_{\mathbb{C}}$ is also in the above-mentioned span. Let $q(M_Z) := \sum_{|\alpha| \leq m} a_\alpha [f(M_Z)]_\alpha$ and let $\xi := \sum_{\beta \in \mathbb{F}_n^+} b_\beta f_\beta \in \mathbb{H}^2(f)$. Note

$$\begin{aligned} P_{\mathbb{C}} q(M_Z)^* \xi = P_{\mathbb{C}} \sum_{|\alpha| \leq m} \overline{a}_\alpha M_{f_\alpha}^* \xi = \sum_{|\alpha| \leq m} \overline{a}_\alpha b_\alpha \\ = \left\langle \xi, \sum_{|\alpha| \leq m} a_\alpha f_\alpha \right\rangle = \langle \xi, q(M_Z) 1 \rangle. \end{aligned}$$

Consequently, if $r(M_Z) := \sum_{|\gamma| \leq p} c_\gamma [f(M_Z)]_\gamma$, then

$$r(M_Z) P_{\mathbb{C}} q(M_Z)^* \xi = \langle \xi, q(M_Z) 1 \rangle \, r(M_Z) 1, \tag{3.1}$$

which shows that $r(M_Z) P_{\mathbb{C}} q(M_Z)^*$ is a rank one operator in $B(\mathbb{H}^2(f))$. Taking into account that the vectors of the form $\sum_{|\alpha| \leq m} a_\alpha [f(M_Z)]_\alpha 1$, where $m \in \mathbb{N}$, $a_\alpha \in \mathbb{C}$, are dense in $\mathbb{H}^2(f)$, and using relation (3.1), we deduce that all compact operators in $B(\mathbb{H}^2(f))$ are in $\overline{\text{span}}\{M_{Z_\alpha} M_{Z_\beta}^* : \alpha, \beta \in \mathbb{F}_n^+\}$. The proof is complete. $\square$

The next result, is a Wold type decomposition for nondegenerate $*$-representations of the $C^*$-algebra $C^*(M_{Z_1}, \ldots, M_{Z_n})$.

**Theorem 3.2.** *Let $f = (f_1, \ldots, f_n)$ be an $n$-tuple of formal power series in the class $\mathcal{M}^{||}$ and let $(M_{Z_1}, \ldots, M_{Z_n})$ be the universal model associated with $\mathbb{B}_f$. If $\pi : C^*(M_{Z_1}, \ldots, M_{Z_n}) \to B(\mathcal{K})$ is a nondegenerate $*$-representation of $C^*(M_{Z_1}, \ldots, M_{Z_n})$ on a separable Hilbert space $\mathcal{K}$, then $\pi$ decomposes into a direct sum*

$$\pi = \pi_0 \oplus \pi_1 \quad on \quad \mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1,$$

*where $\pi_0$, $\pi_1$ are disjoint representations of $C^*(M_{Z_1}, \ldots, M_{Z_n})$ on the Hilbert spaces*

$$\mathcal{K}_0 := \overline{\text{span}} \left\{ \pi(M_{Z_\alpha}) \left( I - \sum_{i=1}^n f_i(\pi(M_{Z_1}), \ldots, \pi(M_{Z_n})) f_i(\pi(M_{Z_1}), \ldots, \pi(M_{Z_n}))^* \right) \mathcal{K} : \alpha \in \mathbb{F}_n^+ \right\}$$

*and $\mathcal{K}_1 := \mathcal{K}_0^\perp$, respectively, such that, up to an isomorphism,*

$$\mathcal{K}_0 \simeq \mathbb{H}^2(f) \otimes \mathcal{G}, \quad \pi_0(X) = X \otimes I_{\mathcal{G}}, \quad X \in C^*(M_{Z_1}, \ldots, M_{Z_n}), \tag{3.2}$$

*for some Hilbert space $\mathcal{G}$ with*

$$\dim \mathcal{G} = \dim \left[ \text{range} \left( I - \sum_{i=1}^n f_i(\pi(M_{Z_1}), \ldots, \pi(M_{Z_n})) f_i(\pi(M_{Z_1}), \ldots, \pi(M_{Z_n}))^* \right) \right],$$

*and $\pi_1$ is a $*$-representation which annihilates the compact operators and*

$$\sum_{i=1}^n f_i(\pi_1(M_{Z_1}), \ldots, \pi_1(M_{Z_n})) f_i(\pi_1(M_{Z_1}), \ldots, \pi_1(M_{Z_n}))^* = I_{\mathcal{K}_1}.$$

*Moreover, if $\pi'$ is another nondegenerate $*$-representation of $C^*(M_{Z_1}, \ldots, M_{Z_n})$ on a separable Hilbert space $\mathcal{K}'$, then $\pi$ is unitarily equivalent to $\pi'$ if and only if $\dim \mathcal{G} = \dim \mathcal{G}'$ and $\pi_1$ is unitarily equivalent to $\pi_1'$.*

*Proof.* Since $f = (f_1, \ldots, f_n)$ is an $n$-tuple of formal power series in the class $\mathcal{M}^{||}$, Proposition 3.1 implies that all the compact operators in $B(\mathbb{H}^2(f))$ are contained in $C^*(M_{Z_1}, \ldots, M_{Z_n})$. The standard theory of representations of $C^*$-algebras shows that the representation $\pi$ decomposes into a direct sum $\pi = \pi_0 \oplus \pi_1$ on $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$, where

$$\mathcal{K}_0 := \overline{\text{span}}\{\pi(X)\mathcal{K} : \ X \ \text{ is compact operator in } B(\mathbb{H}^2(f))\} \quad \text{and} \quad \mathcal{K}_1 := \mathcal{K}_0^\perp,$$

and the the representations $\pi_j : C^*(M_{Z_1}, \ldots, M_{Z_n}) \to \mathcal{K}_j$ are defined by $\pi_j(X) := \pi(X)|_{\mathcal{K}_j}$, $j = 0, 1$. The disjoint representations $\pi_0$, $\pi_1$ are such that $\pi_1$ annihilates the compact operators in $B(\mathbb{H}^2(f))$, and $\pi_0$ is uniquely determined by the action of $\pi$ on the ideal of compact operators in $B(\mathbb{H}^2(f))$. Taking into account that every representation of the compact operators on $\mathbb{H}^2(f)$ is equivalent to a multiple of the identity representation, we deduce relation (3.2). Using the proof of Theorem 3.1, we deduce that

$$\begin{aligned}\mathcal{K}_0 &:= \overline{\text{span}}\{\pi(X)\mathcal{K} :\ X \text{ is compact operator in } B(\mathbb{H}^2(f))\}\\ &= \overline{\text{span}}\{\pi(M_{Z_\alpha} P_{\mathbb{C}} M_{Z_\beta}^*)\mathcal{K} :\ \alpha, \beta \in \mathbb{F}_n^+\}\\ &= \overline{\text{span}}\left\{\pi(M_{Z_\alpha})\left(I - \sum_{i=1}^n f_i(\pi(M_{Z_1}), \ldots, \pi(M_{Z_n})) f_i(\pi(M_{Z_1}), \ldots, \pi(M_{Z_n}))^*\right)\mathcal{K} :\ \alpha \in \mathbb{F}_n^+\right\}.\end{aligned}$$

Now, since $P_{\mathbb{C}} = I - \sum_{i=1}^n f_i(M_{Z_1}, \ldots, M_{Z_n}) f_i(M_{Z_1}, \ldots, M_{Z_n})^*$ is a rank one projection in the $C^*$-algebra $C^*(M_{Z_1}, \ldots, M_{Z_n})$ , we have

$$\sum_{i=1}^n f_i(\pi_1(M_{Z_1}), \ldots, \pi_1(M_{Z_n})) f_i(\pi_1(M_{Z_1}), \ldots, \pi_1(M_{Z_n}))^* = I_{\mathcal{K}_1}$$

and

$$\dim \mathcal{G} = \dim\left[\text{range}\, \pi(P_{\mathbb{C}})\right] = \dim\left[\text{range}\left(I - \sum_{i=1}^n f_i(\pi(M_{Z_1}), \ldots, \pi(M_{Z_n})) f_i(\pi(M_{Z_1}), \ldots, \pi(M_{Z_n}))^*\right)\right].$$

To prove the last part of the theorem, we recall that, according to the standard theory of representations of $C^*$-algebras, $\pi$ and $\pi'$ are unitarily equivalent if and only if $\pi_0$ and $\pi_0'$ (resp. $\pi_1$ and $\pi_1'$) are unitarily equivalent. On the other hand, we proved in [27] that the $C^*$-algebra $C^*(M_{Z_1}, \ldots, M_{Z_n})$ is irreducible and, consequently, the $n$-tuples $(M_{Z_1} \otimes I_{\mathcal{K}}, \ldots, M_{Z_n} \otimes I_{\mathcal{K}})$ is unitarily equivalent to $(M_{Z_1} \otimes I_{\mathcal{K}'}, \ldots, M_{Z_n} \otimes I_{\mathcal{K}'})$ if and only if $\dim \mathcal{K} = \dim \mathcal{K}'$. Hence, we conclude that $\dim \mathcal{G} = \dim \mathcal{G}'$ and complete the proof. $\square$

We introduce the class $\mathcal{M}_{rad}^b$ of all formal power series $f = (f_1, \ldots, f_n)$ with the property that there is $\delta \in (0, 1)$ such that $rf \in \mathcal{M}^b$ for any $r \in (\delta, 1]$. We remark that in all the examples presented in [27], the corresponding $n$-tuples $f = (f_1, \ldots, f_n)$ are in the class $\mathcal{M}_{rad}^{||} \subset \mathcal{M}_{rad}^b$. Moreover, the $n$-tuple of polynomials with property $(\mathcal{A})$ are also in the class $\mathcal{M}_{rad}^b$. The *coisometric* part of $\mathbb{B}_f(\mathcal{H})$ is defined as the set

$$\mathbb{B}_f^c(\mathcal{H}) := \{X = (X_1, \ldots, X_n) \in \mathbb{B}_f(\mathcal{H}) :\ \sum_{i=1}^n f_i(X) f_i(X)^* = I\}.$$

**Proposition 3.3.** *Let $f = (f_1, \ldots, f_n)$ be an $n$-tuple of formal power series with the model property and let $g = (g_1, \ldots, g_n)$ be its inverse with respect to the composition. If $f$ satisfies either one of the following conditions:*

(i) $f \in \mathcal{M}_{rad}^b$;
(ii) $f \in \mathcal{M}_{rad} \cap \mathcal{M}^{||}$;
(iii) $f \in \mathcal{M}^{||}$ *and* $g \in \mathcal{A}_n$,

*then $\mathbb{B}_f(\mathcal{H}) = g\left([B(\mathcal{H})^n]_1^-\right)$. Moreover, in this case, $g : [B(\mathcal{H})^n]_1^- \to \mathbb{B}_f(\mathcal{H})$ is a bijection with inverse $f : \mathbb{B}_f(\mathcal{H}) \to [B(\mathcal{H})^n]_1^-$. In particular, $g([B(\mathcal{H})^n]_1^c) = \mathbb{B}_f^c(\mathcal{H})$.*

*Proof.* Assume that condition $(i)$ holds. Since $\mathbb{B}_f(\mathcal{H}) \subseteq g\left([B(\mathcal{H})^n]_1^-\right)$, it remains to prove the reverse inclusion. Let $Y := g(X)$ with $X = (X_1, \ldots, X_n) \in [B(\mathcal{H})^n]_1^-$. Assume that $f_i := \sum_{\alpha \in \mathbb{F}_n^+} c_\alpha^{(i)} Z_\alpha$, $i = 1, \ldots, n$. Since $f = (f_1, \ldots, f_n)$ has the radial approximation property, $g_i := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha^{(i)} Z_\alpha$ is a free holomorphic function on $[B(\mathcal{H})^n]_\gamma$ for some $\gamma > 1$. Moreover, there is $\delta \in (0, 1)$ with the property that for any $r \in (\delta, 1]$, the series $g_i(\frac{1}{r}S) := \sum_{k=0}^\infty \sum_{|\alpha|=k} \frac{a_\alpha^{(i)}}{r^{|\alpha|}} S_\alpha$ is convergent in the operator norm topology and represents an element in the noncommutative disc algebra $\mathcal{A}_n$, and

$$\frac{1}{r} S_j = f_j\left(g_1\left(\frac{1}{r}S\right), \ldots, g_n\left(\frac{1}{r}S\right)\right), \qquad j \in \{1, \ldots, n\},\ r \in (\delta, 1],$$

where $g(\frac{1}{r}S)$ is in the SOT-convergence (or radial SOT-convergence) of $f$ and $S=(S_1,\ldots,S_n)$ is the $n$-tuple of left creation operators on the Fock space $F^2(H_n)$. Since $f\in\mathcal{M}^b_{rad}$, we deduce that one of the following conditions holds:

(a) $\frac{1}{r}S_i=\text{SOT-}\lim_{m\to\infty}\sum_{|\alpha|\leq m}c_\alpha^{(i)}g_\alpha\left(\frac{1}{r}S\right)$ and

$$\sup_{m\in\mathbb{N}}\left\|\sum_{|\alpha|\leq m}c_\alpha^{(i)}g_\alpha\left(\frac{1}{r}S\right)\right\|<\infty,\qquad i=1,\ldots,n;$$

(b) $\frac{1}{r}S_i=\text{SOT-}\lim_{\gamma\to 1}\sum_{k=0}^\infty\sum_{|\alpha|=k}c_\alpha^{(i)}\gamma^{|\alpha|}g_\alpha\left(\frac{1}{r}S\right)$ and

$$\sup_{\gamma\in[0,1)}\left\|\sum_{k=0}^\infty\sum_{|\alpha|=k}c_\alpha^{(i)}\gamma^{|\alpha|}g_\alpha\left(\frac{1}{r}S\right)\right\|<\infty,\qquad i=1,\ldots,n.$$

As in the proof of Proposition 2.2, using the fact that the noncommutative Poisson transform $P_{rX}$, $r\in(\delta,1)$, is $SOT$-continuous on bounded sets, we deduce that that $X_j=f_j(g(X))$ for $j=1,\ldots,n$. This also shows that $g$ is one-to-one on $[B(\mathcal{H})^n]_1^-$. On the other hand, the relation above implies $Y=g(X)=g(f(g(X)))=g(f(Y))$ and $\|f(Y)\|\leq 1$, which shows that $Y\in\mathbb{B}_f(\mathcal{H})$. Therefore, $\mathbb{B}_f(\mathcal{H})=g\left([B(\mathcal{H})^n]_1^-\right)$ and $f$ is one-to-one on $\mathbb{B}_f(\mathcal{H})$. Hence, we also deduce that $g([B(\mathcal{H})^n]_1^c)=\mathbb{B}_f^c(\mathcal{H})$. Similarly, one can prove this proposition when condition $(ii)$ or $(iii)$ holds. The proof is complete. ☐

**Theorem 3.4.** *Let $f=(f_1,\ldots,f_n)$ be an $n$-tuple of formal power series with the model property and let $(M_{Z_1},\ldots,M_{Z_n})$ be the universal model associated with $\mathbb{B}_f$. If $f\in\mathcal{M}^{||}$ and $\pi$ is a $*$-representation of $C^*(M_{Z_1},\ldots,M_{Z_n})$, then*

$$[f_1(\pi(M_{Z_1}),\ldots,\pi(M_{Z_n})),\ldots,f_n(\pi(M_{Z_1}),\ldots,\pi(M_{Z_n}))]$$

*is a row isometry.*

*Conversely, if $f\in\mathcal{M}^b_{rad}$ or $f\in\mathcal{M}_{rad}\cap\mathcal{M}^{||}$, and $[W_1,\ldots,W_n]\in B(\mathcal{K})^n$ is a row isometry, then there is a unique $*$-representation $\pi:C^*(M_{Z_1},\ldots,M_{Z_n})\to B(\mathcal{K})$ such that $\pi(M_{Z_i})=g_i(W_1,\ldots,W_n)$, $i=1,\ldots,n$, where $g=(g_1,\ldots,g_n)$ is the inverse of $f$ with respect to the composition.*

*Proof.* Let $f_i$ have the representation $f_i=\sum_{\alpha\in\mathbb{F}_n^+}c_\alpha^{(i)}Z_\alpha$. Assuming that $f\in\mathcal{M}^{||}$, we deduce that $f_i(M_{Z_1},\ldots,M_{Z_n})=\sum_{k=o}^\infty\sum_{|\alpha|=k}c_\alpha^{(i)}M_{Z_\alpha}$ or $f_i(M_{Z_1},\ldots,M_{Z_n})=\lim_{r\to 1}\sum_{k=0}^\infty\sum_{|\alpha|=k}r^{|\alpha|}c_\alpha^{(i)}M_{Z_\alpha}$ where the convergence is in the operator norm topology. In either case, if $\pi:C^*(M_{Z_1},\ldots,M_{Z_n})\to B(\mathcal{K})$ is a $*$-representation, we have

$$\pi(f_i(M_{Z_1},\ldots,M_{Z_n}))=f_i(\pi(M_{Z_1}),\ldots,\pi(M_{Z_n})),\qquad i=1,\ldots,n,$$

and, taking into account that $f_i(M_{Z_1},\ldots,M_{Z_n})=M_{f_i}$, we obtain

$$f_i(\pi(M_{Z_1}),\ldots,\pi(M_{Z_n}))^*f_i(\pi(M_{Z_1}),\ldots,\pi(M_{Z_n}))=\pi(M_{f_i}^*M_{f_i})=\delta_{ij}I_\mathcal{K}$$

for any $i,j=1,\ldots,n$. Therefore, $[f_1(\pi(M_{Z_1}),\ldots,\pi(M_{Z_n})),\ldots,f_n(\pi(M_{Z_1}),\ldots,\pi(M_{Z_n}))]$ is a row isometry.

Conversely, assume that $f\in\mathcal{M}^b_{rad}$ or $f\in\mathcal{M}_{rad}\cap\mathcal{M}^{||}$ and $[W_1,\ldots,W_n]\in B(\mathcal{K})^n$ is a row isometry. Let $g=(g_1,\ldots,g_n)$ be the inverse of $f$ with respect to the composition and let $g_i:=\sum_{\alpha\in\mathbb{F}_n^+}a_\alpha^{(i)}Z_\alpha$. Since $f$ has the radial approximation property, we deduce that $g$ is a free holomorphic function on $[B(\mathcal{H})^n]_\gamma$ for some $\gamma>1$. Consequently, $g_i(W_1,\ldots,W_n)=\sum_{k=0}^\infty\sum_{|\alpha|=k}a_\alpha^{(i)}W_\alpha$, where the convergence is in the operator norm topology. Applying Proposition 3.3, we deduce that $(g_1(W),\ldots,g_n(W))\in\mathbb{B}_f(\mathcal{K})$ and $f_i(g_1(W),\ldots,g_n(W))=W_i$ for $i=1,\ldots,n$.

According to [27], since $f=(f_1,\ldots,f_n)$ has the radial approximation property and $(g_1(W),\ldots,g_n(W))\in\mathbb{B}_f(\mathcal{K})$, there is a unique unital completely contractive linear map

$$\pi:C^*(M_{Z_1},\ldots,M_{Z_n})\to B(\mathcal{K})$$

such that

$$\pi(M_{Z_\alpha} M_{Z_\beta}^*) = g_\alpha(W) g_\beta(W)^*, \qquad \alpha, \beta \in \mathbb{F}_n^+. \tag{3.3}$$

On the other hand, since $[W_1, \ldots, W_n] \in B(\mathcal{K})^n$ is a row isometry, we deduce that

$$\begin{aligned}
\langle g_i(W)^* g_j(W) x, y \rangle &= \left\langle \sum_{k=0}^\infty \sum_{|\beta|=k} a_\beta^{(j)} W_\beta x, \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha^{(i)} W_\alpha y \right\rangle \\
&= \lim_{m\to\infty} \left\langle \sum_{|\alpha|\leq m} \sum_{k=0}^\infty \sum_{|\beta|=k} \overline{a_\alpha^{(i)}} a_\beta^{(j)} W_\alpha^* W_\beta x, y \right\rangle \\
&= \lim_{m\to\infty} \left\langle \sum_{|\alpha|\leq m} \sum_{k=0}^\infty \sum_{|\beta|=k} \overline{a_\alpha^{(i)}} a_\beta^{(j)} \delta_{\alpha\beta} x, y \right\rangle \\
&= \lim_{m\to\infty} \sum_{|\alpha|\leq m} \overline{a_\alpha^{(i)}} a_\alpha^{(j)} \langle x, y \rangle \\
&= \sum_{k=0}^\infty \sum_{|\alpha|=k} \overline{a_\alpha^{(i)}} a_\alpha^{(j)} \langle x, y \rangle \\
&= \langle Z_j, Z_i \rangle_{\mathbb{H}^2(f)} \langle x, y \rangle
\end{aligned}$$

for any $x, y \in \mathcal{K}$. Hence, and using the fact that

$$M_{Z_i}^* M_{Z_j} = \langle Z_j, Z_i \rangle_{\mathbb{H}^2(f)} I_{\mathbb{H}^2(f)}, \qquad i, j \in \{1, \ldots, n\},$$

we deduce that $\pi(M_{Z_i}^* M_{Z_j}) = \pi(M_{Z_i})^* \pi(M_{Z_j})$. Therefore, taking into account relation (3.3) and the fact that $C^*(M_{Z_1}, \ldots, M_{Z_n})$ coincides with $\overline{\text{span}}\{M_{Z_\alpha} M_{Z_\beta}^* : \alpha, \beta \in \mathbb{F}_n^+\}$, we conclude that $\pi$ is a $*$-representation of $C^*(M_{Z_1}, \ldots, M_{Z_n})$. The proof is complete. □

**Corollary 3.5.** *Let $f = (f_1, \ldots, f_n)$ be an $n$-tuple of formal power series in the set $\mathcal{M}_{rad} \cap \mathcal{M}^{||}$. Then any $*$-representation $\pi : C^*(M_{Z_1}, \ldots, M_{Z_n}) \to B(\mathcal{K})$ is generated by a row isometry $[W_1, \ldots, W_n]$, $W_i \in B(\mathcal{K})$, such that*

$$\pi(M_{Z_i}) = g_i(W_1, \ldots, W_n), \qquad i = 1, \ldots, n.$$

We remark that, in the particular case when $f \in \mathcal{M}_{rad} \cap \mathcal{M}^{||}$, one can use Corollary 3.5 and the Wold decomposition for isometries with orthogonal ranges [10], to provide another proof of Theorem 3.2.

Let $T := (T_1, \ldots, T_n) \in \mathbb{B}_f(\mathcal{H})$. We say that an $n$-tuple $V := (V_1, \ldots, V_n)$ of operators on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ is a minimal dilation of $T$ if the following properties are satisfied:

(i) $(V_1, \ldots, V_n) \in \mathbb{B}_f(\mathcal{K})$;
(ii) there is a $*$-representation $\pi : C^*(M_{Z_1}, \ldots, M_{Z_n}) \to B(\mathcal{K})$ such that $\pi(M_{Z_i}) = V_i$, $i = 1, \ldots, n$;
(iii) $V_i^*|_{\mathcal{H}} = T_i^*$, $i = 1, \ldots, n$;
(iv) $\mathcal{K} = \bigvee_{\alpha \in \mathbb{F}_n^+} V_\alpha \mathcal{H}$.

Without the condition (iv), the $n$-tuple $V$ is called dilation of $T$. We remark that if $f \in \mathcal{M}_{rad} \cap \mathcal{M}^{||}$, then the condition (i) is a consequence of (ii).

**Theorem 3.6.** *Let $f = (f_1, \ldots, f_n)$ be an $n$-tuple of formal power series in the set $\mathcal{M}_{rad}^b$ or $\mathcal{M}_{rad} \cap \mathcal{M}^{||}$ and let $g = (g_1, \ldots, g_n)$ be its inverse with respect to the composition. If $T = (T_1, \ldots, T_n)$ is in the noncommutative domain $\mathbb{B}_f(\mathcal{H})$, then it has a minimal dilation which is unique up to an isomorphism. Moreover, its minimal dilation coincides with $(g_1(W), \ldots, g_n(W))$, where $W = (W_1, \ldots, W_n)$ is the minimal isometric dilation of the row contraction $(f_1(T), \ldots, f_n(T))$ on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$.*

*Proof.* Since $(f_1(T), \ldots, f_n(T))$ is a row contraction, according to [10], there is a minimal isometric dilation $W = (W_1, \ldots, W_n) \in B(\mathcal{K})^n$ with $\mathcal{K} \supseteq \mathcal{H}$. Therefore, we have $W_i^* W_j = \delta_{ij} I_{\mathcal{K}}$, $W_i^*|_{\mathcal{H}} = f_i(T)^*$ for $i = 1, \ldots, n$, and $\mathcal{K} = \bigvee_{\alpha \in \mathbb{F}_n^+} W_\alpha \mathcal{H}$. Applying Proposition 3.3, we deduce that $(g_1(W), \ldots, g_n(W)) \in \mathbb{B}_f(\mathcal{K})$ and $f_i(g_1(W), \ldots, g_n(W)) = W_i$ for $i = 1, \ldots, n$. Since $f \in \mathcal{M}_{rad}$, for each $i = 1, \ldots, n$, we

have that $g_i=\sum_{\alpha\in\mathbb{F}_n^+} a_\alpha^{(i)} Z_\alpha$ is a free holomorphic function on $[B(\mathcal{H})^n]_\gamma$ for some $\gamma>1$. Consequently, $g_i(W)=\sum_{k=0}^\infty\sum_{|\alpha|=k} a_\alpha^{(i)} W_\alpha$ and $g_i(f(T))=\sum_{k=0}^\infty\sum_{|\alpha|=k} a_\alpha^{(i)}[f(T)]_\alpha$ where the convergence is in the operator norm. Hence, and using the fact that $W_i^*|_\mathcal{H}=f_i(T)^*$, we obtain

$$g_i(W)^*|_\mathcal{H}=g_i(f(T))^*|_\mathcal{H},\qquad i=1,\dots,n.$$

Now, using relation $T_i=g_i(f(T))$, we deduce that $g_i(W)^*|_\mathcal{H}=T_i^*|_\mathcal{H}$ for $i=1,\dots,n$. Note also that $\bigvee_{\alpha\in\mathbb{F}_n^+}[g(W)]_\alpha\mathcal{H}\subseteq\bigvee_{\alpha\in\mathbb{F}_n^+}W_\alpha\mathcal{H}=\mathcal{K}$. To prove the reverse inclusion, one can use the relation $f_i(g_1(W),\dots,g_n(W))=W_i$ for $i=1,\dots,n$, and the fact that $(g_1(W),\dots,g_n(W))$ is either in the convergence set $\mathcal{C}_f^{SOT}(\mathcal{K})$ or $\mathcal{C}_f^{rad}(\mathcal{K})$. The fact that there is a $*$-representation $\pi:C^*(M_{Z_1},\dots,M_{Z_n})\to B(\mathcal{K})$ such that $\pi(M_{Z_i})=g_i(W)$ for any $i=1,\dots,n$, follows from Theorem 3.4.

To prove the uniqueness, let $V=(V_1,\dots,V_n)$ and $V'=(V_1',\dots,V_n')$ be two minimal dilations of $T=(T_1,\dots,T_n)\in\mathbb{B}_f(\mathcal{H})$ on the Hilbert space $\mathcal{K}\supseteq\mathcal{H}$ and $\mathcal{K}'\supseteq\mathcal{H}$, respectively. Let $\alpha:=g_{i_1}\cdots g_{i_k}\in\mathbb{F}_n^+$ and $\beta:=g_{j_1}\cdots g_{j_p}\in\mathbb{F}_n^+$. Note that $V_i^*V_j=\pi(M_{Z_i}^*)\pi(M_{Z_j})=\langle Z_j,Z_i\rangle_{\mathbb{H}^2(f)}I_\mathcal{K}$ for $i,j=1,\dots,n$. Consequently, if $k>p$ and $h^{(\beta)},k^{(\alpha)}\in\mathcal{H}$, then

$$\begin{aligned}\left\langle V_\alpha^*V_\beta h^{(\beta)},k^{(\alpha)}\right\rangle&=\left\langle \langle Z_{j_1},Z_{i_1}\rangle\cdots\langle Z_{j_p},Z_{i_p}\rangle V_{i_k}^*\cdots V_{i_{p+1}}^*h^{(\beta)},k^{(\alpha)}\right\rangle\\&=\langle Z_{j_1},Z_{i_1}\rangle\cdots\langle Z_{j_p},Z_{i_p}\rangle\left\langle T_{i_k}^*\cdots T_{i_{p+1}}^*h^{(\beta)},k^{(\alpha)}\right\rangle.\end{aligned}$$

When $p>k$, we obtain

$$\left\langle V_\alpha^*V_\beta h^{(\beta)},k^{(\alpha)}\right\rangle=\langle Z_{j_1},Z_{i_1}\rangle\cdots\langle Z_{j_k},Z_{i_k}\rangle\left\langle h^{(\beta)},T_{i_{k+1}}\cdots T_{i_p}k^{(\alpha)}\right\rangle,$$

and, if $k=p$, we have $\left\langle V_\alpha^*V_\beta h^{(\beta)},k^{(\alpha)}\right\rangle=\langle Z_{j_1},Z_{i_1}\rangle\cdots\langle Z_{j_k},Z_{i_k}\rangle\left\langle h^{(\beta)},k^{(\alpha)}\right\rangle$. Similar relations hold for the minimal dilation $V'=(V_1',\dots,V_n')$. Hence, and taking into account that the dilations are minimal, i.e., $\mathcal{K}=\bigvee_{\alpha\in\mathbb{F}_n^+}V_\alpha\mathcal{H}$ and $\mathcal{K}'=\bigvee_{\alpha\in\mathbb{F}_n^+}V_\alpha'\mathcal{H}$, one can easily see that there is a unitary operator $U:\mathcal{K}\to\mathcal{K}'$ such that $U\left(\sum_{|\alpha|\le m}V_\alpha h^{(\alpha)}\right)=\sum_{|\alpha|\le m}V_\alpha'h^{(\alpha)}$ for any $h^{(\alpha)}\in\mathcal{H}$, $|\alpha|\le m$, and $m\in\mathbb{N}$. Consequently, we deduce that $UV_i=V_i'U$ for any $i=1,\dots,n$. The proof is complete. $\square$

Using Theorem 3.6 and Theorem 3.4, we deduce the following result.

**Corollary 3.7.** *Let $f=(f_1,\dots,f_n)$ be an $n$-tuple of formal power series with the model property and let $T=(T_1,\dots,T_n)\in\mathbb{B}_f(\mathcal{H})$.*

(i) *If $f\in\mathcal{M}_{rad}^b$ or $f\in\mathcal{M}_{rad}\cap\mathcal{M}^{||}$, then $(g_1(W),\dots,g_n(W))$ is a dilation of $T=(T_1,\dots,T_n)$, where $W=(W_1,\dots,W_n)$ is an isometric dilation of the row contraction $(f_1(T),\dots,f_n(T))$.*

(ii) *If $f\in\mathcal{M}^{||}$ and $V=(V_1,\dots,V_n)\in\mathbb{B}_f(\mathcal{K})$ is a dilation of $T$, then $(f_1(V),\dots,f_n(V))$ is an isometric dilation of $(f_1(T),\dots,f_n(T))$.*

We remark that under the conditions and notations of Theorem 2.5 (see also the proof), if $f\in\mathcal{M}_{rad}^b$, $\Theta$ is purely contractive, and

$$\overline{\Delta_\Theta(\mathbb{H}^2(f)\otimes\mathcal{E}_*)}=\overline{\Delta_\Theta(\vee_{i=1}^n M_{f_i}(\mathbb{H}^2(f)\otimes\mathcal{E}_*))},$$

then, considering $\mathbf{H}$ as a subspace of

$$\mathbf{K}:=(\mathbb{H}^2(f)\otimes\mathcal{E})\oplus\overline{\Delta_\Theta(\mathbb{H}^2(f)\otimes\mathcal{E}_*))},$$

one can prove that the sequence of operators $\mathbf{V}:=(\mathbf{V}_1,\dots,\mathbf{V}_n)$ defined on $\mathbf{K}$ by

$$\mathbf{V}_i:=(M_{Z_i}\otimes I_\mathcal{E})\oplus D_i,\qquad i=1,\dots,n,$$

is the minimal dilation of $\mathbf{T}:=(\mathbf{T}_1,\dots,\mathbf{T}_n)\in\mathbb{B}_f^{cnc}(\mathbf{H})$. Indeed, according to Theorem 4.1 from [11], the multi-analytic operator $\Psi$ coincides with the characteristic function $\Theta_\mathbf{A}$ of the row contraction $\mathbf{A}=[\mathbf{A}_1,\dots,\mathbf{A}_n]$. Moreover, the $n$-tuple $\mathbf{W}:=[\mathbf{W}_1,\dots,\mathbf{W}_n]$ defined on $\widetilde{\mathcal{K}}$ is the minimal isometric dilation of $\mathbf{A}$. Consequently, $\mathbf{W}:=[\mathbf{W}_1,\dots,\mathbf{W}_n]$ is a row isometry, $\mathbf{W}_i^*|_{\widetilde{\mathcal{H}}}=\mathbf{A}_i^*$ for $i=1,\dots,n$, and $\widetilde{\mathcal{K}}=\bigvee_{\alpha\in\mathbb{F}_n^+}\mathbf{W}_\alpha\widetilde{\mathcal{H}}$. On the other hand, since $f=(f_1,\dots,f_n)$ has the radial approximation property, we have $g_i(\mathbf{W}_1,\dots,\mathbf{W}_n)=g_i(S_1,\dots,S_n)\oplus g_i(C_1,\dots,C_n)$, where the convergence defining these operators is in

the operator norm topology. Now, it easy to see that $g_i(\mathbf{W}_1,\ldots,\mathbf{W}_n)^*|_{\widetilde{\mathcal{H}}} = g_i(\mathbf{A}_1,\ldots,\mathbf{A}_n)^* = \mathbb{T}_i^*$ and $\widetilde{\mathcal{K}} = \bigvee_{\alpha\in\mathbb{F}_n^+} g_\alpha(\mathbf{W}_1,\ldots,\mathbf{W}_n)\widetilde{\mathcal{H}}$. Note also that $\mathbf{V}_i = \Gamma^{-1}g_i(\mathbf{W}_1,\ldots,\mathbf{W}_n)\Gamma$, $i=1,\ldots,n$. Taking into account that $\Gamma = (U\otimes\mathcal{E})\oplus(U\otimes I_{\mathcal{E}_*})$ is a unitary operator with the property that $\Gamma(\mathbf{H}) = \widetilde{\mathcal{H}}$, we deduce that $\mathbf{V_i}^*|_{\mathbf{H}} = \mathbf{T}_i$, $i=1,\ldots,n$ and $\mathbf{K} = \bigvee_{\alpha\in\mathbb{F}_n^+}\mathbf{V}_\alpha\mathbf{H}$. Using Theorem 3.6, we deduce that $(\mathbf{V}_1,\ldots,\mathbf{V}_n)$ is the minimal dilation of $(\mathbf{T}_1,\ldots,\mathbf{T}_n)$, which proves our assertion.

In what follows, we provide a commutant lifting theorem for the noncommutative domains $\mathbb{B}_f$.

**Theorem 3.8.** *Let $f=(f_1,\ldots,f_n)$ be an $n$-tuple of formal power series in the set $\mathcal{M}_{rad}\cap\mathcal{M}^{||}$. Let $T=(T_1,\ldots,T_n)\in\mathbb{B}_f(\mathcal{H})$ and $T'=(T_1',\ldots,T_n')\in\mathbb{B}_f(\mathcal{H}')$, and let $V=(V_1,\ldots,V_n)\in\mathbb{B}_f(\mathcal{K})$ and $V'=(V_1',\ldots,V_n')\in\mathbb{B}_f(\mathcal{K}')$ be dilations of $T$ and $T'$, respectively. If $X:\mathcal{H}\to\mathcal{H}'$ is bounded operator satisfying the intertwining relations $XT_i = T_i'X$ for any $i=1,\ldots n$, then there exists a bounded operator $Y:\mathcal{K}\to\mathcal{K}'$ with the following properties:*

(i) *$YV_i = V_i'Y$ for any $i=1,\ldots n$;*
(ii) *$Y^*|_{\mathcal{H}} = X^*$ and $\|X\|=\|Y\|$.*

*Proof.* According to Corollary 3.7, since $f\in\mathcal{M}_{rad}\cap\mathcal{M}^{||}$, there are isometric dilations of any $n$-tuple $T=(T_1,\ldots,T_n)\in\mathbb{B}_f(\mathcal{H})$. Let $V=(V_1,\ldots,V_n)\in\mathbb{B}_f(\mathcal{K})$ be a dilation of $T$. Then $V_i^*|_{\mathcal{H}} = T_i^*$, $i=1,\ldots,n$, and there is a $*$-representation $\pi: C^*(M_{Z_1},\ldots,M_{Z_n})\to B(\mathcal{K})$ such that $\pi(M_{Z_i}) = V_i$, $i=1,\ldots,n$. Let $f_i$ have the representation $f_i = \sum_{\alpha\in\mathbb{F}_n^+} c_\alpha^{(i)} Z_\alpha$. Taking into account that $f\in\mathcal{M}^{||}$, we deduce that $f_i(M_{Z_1},\ldots,M_{Z_n}) = \sum_{k=o}^\infty\sum_{|\alpha|=k} c_\alpha^{(i)}M_{Z_\alpha}$ or $f_i(M_{Z_1},\ldots,M_{Z_n}) = \lim_{r\to 1}\sum_{k=0}^\infty\sum_{|\alpha|=k} r^{|\alpha|}c_\alpha^{(i)}M_{Z_\alpha}$, where the convergence is in the operator norm topology. Consequently, we have

$$f_i(V_1,\ldots,V_n) = f_i(\pi(M_{Z_1}),\ldots,\pi(M_{Z_n})) = \pi(f_i(M_{Z_1},\ldots,M_{Z_n}), \qquad i=1,\ldots,n,$$

and

$$\begin{aligned} f_i(V_1,\ldots,V_n)^* f_i(V_1,\ldots,V_n) &= \pi(f_i(M_{Z_1},\ldots,M_{Z_n}^* f_i(M_{Z_1},\ldots,M_{Z_n})) \\ &= \pi(M_{f_i}^* M_{f_i}) = \delta_{ij} I_{\mathcal{K}} \end{aligned}$$

for any $i,j=1,\ldots,n$. Since $V_i^*|_{\mathcal{H}} = T_i^*$, we also have $f_i(V_1,\ldots,V_n)^*|_{\mathcal{H}} = f_i(T_1,\ldots,T_n)$ for any $i=1,\ldots,n$. This shows that $(f_1(V),\ldots,f_n(V))$ is an isometric dilation of $(f_1(T),\ldots,f_n(T))$. A similar results holds if $V'=(V_1',\ldots,V_n')\in\mathbb{B}_f(\mathcal{K}')$ is a dilation of $T'=(T_1',\ldots,T_n')\in\mathbb{B}_f(\mathcal{H}')$. Now, we assume that $X:\mathcal{H}\to\mathcal{H}'$ is a bounded operator such that $XT_i = T_i'X$ for $i=1,\ldots,n$. Hence, we deduce that $Xf_i(T) = f_i(T')X$, $i=1,\ldots,n$. Applying the noncommutative commutant lifting theorem for row contractions [10], we find an operator $Y:\mathcal{K}\to\mathcal{K}'$ such that $Yf_i(V) = f_i(V')Y$ for $i=1,\ldots,n$, $Y^*|_{\mathcal{H}'} = X^*$, and $\|Y\|=\|X\|$. Consequently, we have $Yg_i(f(V)) = g_i(f(V'))Y$, $i=1,\ldots,n$. Since $g_i(f(V)) = V_i$ and $g_i(f(V')) = V'$, we conclude that $YV_i = V_i'Y$ for any $i=1,\ldots,n$. The proof is complete. $\square$

## 4. Noncommutative varieties, constrained characteristic functions, and operator models

We present operator models, in terms of constrained characteristic functions, for $n$-tuples of operators in noncommutative varieties $\mathcal{V}_{f,J}^{cnc}(\mathcal{H})$ associated with WOT-closed two-sided ideals $J$ of the Hardy algebra $H^\infty(\mathbb{B}_f)$. This is used to show that the constrained characteristic function $\Theta_{f,T,J}$ is a complete unitary invariant for $\mathcal{V}_{f,J}^{cnc}(\mathcal{H})$.

Let $f=(f_1,\ldots,f_n)$ be an $n$-tuple of formal power series with the model property. The noncommutative Hardy algebra $H^\infty(\mathbb{B}_f)$ is the WOT-closure of all noncommutative polynomials in $M_{Z_1},\ldots,M_{Z_n}$ and the identity. According to [27], $J$ is a WOT-closed two-sided ideal of $H^\infty(\mathbb{B}_f)$ if and only if there is a WOT-closed two-sided ideal $\mathcal{I}$ of the noncommutative analytic Toeplitz algebra $F_n^\infty$ such that

$$J = \{\varphi(f(M_Z)) :\ \varphi\in\mathcal{I}\}.$$

We mention that if $\varphi(S_1,\dots,S_n)\in F_n^\infty$ has the Fourier representation $\varphi(S_1,\dots,S_n)=\sum_{\alpha\in\mathbb{F}_n^+} c_\alpha S_\alpha$, then

$$\varphi(f(M_Z)) := \text{SOT-}\lim_{r\to 1}\sum_{k=0}^\infty \sum_{|\alpha|=k} c_\alpha r^{|\alpha|}[f(M_Z)]_\alpha$$

exists. Denote by $H^\infty(\mathcal{V}_{f,J})$ the WOT-closed algebra generated by the operators $B_i := P_{\mathcal{N}_{f,J}} M_{Z_i}|_{\mathcal{N}_{f,J}}$, for $i=1,\dots,n$, and the identity, where

$$\mathcal{N}_{f,J} := \mathbb{H}^2(f)\ominus \mathcal{M}_{f,J} \quad \text{and} \quad \mathcal{M}_{f,J} := \overline{J\mathbb{H}^2(f)}.$$

We recall that the map

$$\Gamma : H^\infty(\mathbb{B}_f)/J \to B(\mathcal{N}_{f,J}) \quad \text{defined by} \quad \Gamma(\varphi+J) = P_{\mathcal{N}_{f,J}}\varphi|_{\mathcal{N}_{f,J}}$$

is a completely isometric representation. Since the set of all polynomials in $M_{Z_1},\dots,M_{Z_n}$ and the identity is WOT-dense in $H^\infty(\mathbb{B}_f)$, we can conclude that $P_{\mathcal{N}_{f,J}}H^\infty(\mathbb{B}_f)|_{\mathcal{N}_{f,J}}$ is a WOT-closed subalgebra of $B(\mathcal{N}_{f,J})$ and, moreover, $H^\infty(\mathcal{V}_{f,J}) = P_{\mathcal{N}_{f,J}}H^\infty(\mathbb{B}_f)|_{\mathcal{N}_{f,J}}$.

Given an $n$-tuple $T=(T_1,\dots,T_n)\in\mathbb{B}_f(\mathcal{H})$, we consider the defect operator

$$\Delta_{f,T} := \left(I-\sum_{j=1}^n f_j(T)f_j(T)^*\right)^{1/2}$$

and the defect space $\mathcal{D}_{f,T} := \overline{\Delta_{f,T}\mathcal{H}}$. Define the noncommutative Poisson kernel $K_{f,T}:\mathcal{H}\to\mathbb{H}^2(f)\otimes\mathcal{D}_{f,T}$ by setting

$$K_{f,T}h := \sum_{\alpha\in\mathbb{F}_n^+} f_\alpha\otimes \Delta_{f,T}[f(T)]_\alpha^* h, \qquad h\in\mathcal{H}. \tag{4.1}$$

Let $J\neq H^\infty(\mathbb{B}_f)$ be a WOT-closed two-sided ideal of $H^\infty(\mathbb{B}_f)$. The *constrained Poisson kernel* associated with $f$, $T$, and $J$ is the operator $K_{f,T,J}:\mathcal{H}\to\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}$ defined by

$$K_{f,T,J} := (P_{\mathcal{N}_{f,J}}\otimes I_{\mathcal{D}_{f,T}})K_{f,T}.$$

We remark that if $\psi=\varphi(f(M_Z))$ for some $\varphi(S_1,\dots,S_n)=\sum_{\alpha\in\mathbb{F}_n^+}c_\alpha S_\alpha$ in the noncommutative analytic Toeplitz algebra $F_n^\infty$, and $T:=(T_1,\dots,T_n)$ is a c.n.c. $n$-tuple in $\mathbb{B}_f(\mathcal{H})$, then $f(T)=(f_1(T),\dots,f_n(T))$ is a c.n.c. row contraction and, due to the $F_n^\infty$-functional calculus [13], the limit

$$\psi(T_1,\dots,T_n) := \text{SOT-}\lim_{r\to 1}\sum_{k=0}^\infty\sum_{|\alpha|=k} c_\alpha r^{|\alpha|}[f(T)]_\alpha$$

exists. Therefore, we can talk about an $H^\infty(\mathbb{B}_f)$-functional calculus for the $n$-tuples of operators in $\mathbb{B}_f^{cnc}(\mathcal{H})$. We introduce the noncommutative variety $\mathcal{V}_{f,J}^{cnc}(\mathcal{H})\subset\mathbb{B}_f(\mathcal{H})$ defined by

$$\mathcal{V}_{f,J}^{cnc}(\mathcal{H}) := \left\{(T_1,\dots,T_n)\in\mathbb{B}_f^{cnc}(\mathcal{H}):\ \psi(T_1,\dots,T_n)=0 \ \text{ for any } \ \psi\in J\right\}.$$

Note that $\psi(B_1,\dots,B_n)=0$ for any $\psi\in J$. The $n$-tuple $B=(B_1,\dots,B_n)\in\mathcal{V}_{f,J}^{cnc}(\mathcal{N}_{f,J})$ will play the role of universal model for the noncommutative variety $\mathcal{V}_{f,J}^{cnc}$.

**Proposition 4.1.** *Let $f=(f_1,\dots,f_n)$ be an $n$-tuple of formal power series with the model property and let $J\neq H^\infty(\mathbb{B}_f)$ be a WOT-closed two-sided ideal of $H^\infty(\mathbb{B}_f)$. If $T:=(T_1,\dots,T_n)$ is an $n$-tuple in $\mathcal{V}_{f,J}^{cnc}(\mathcal{H})$, then*

$$K_{f,T,J}T_i^* = (B_i^*\otimes I_{\mathcal{D}_{f,T}})K_{f,T,J}, \qquad i=1,\dots,n,$$

*and*

$$K_{f,T,J}^*K_{f,T,J} = I_{\mathcal{H}} - \text{SOT-}\lim_{q\to\infty}\sum_{\alpha\in\mathbb{F}_n,\,|\alpha|=q}[f(T)]_\alpha[f(T)]_\alpha^*,$$

*where $K_{f,T,J}$ is the constrained Poisson kernel associated with $f$, $T$, and $J$.*

*Proof.* The fact that

$$K_{f,T}T_i^* = (M_{Z_i}^* \otimes I_{\mathcal{D}_{f,T}})K_{f,T}, \qquad i = 1, \ldots, n, \tag{4.2}$$

was proved in Theorem 4.1 from [27]. Now, we show that

$$K_{f,T,J}T_i^* = (B_i^* \otimes I_{\mathcal{D}_{f,T}})K_{f,T,J}, \qquad i = 1, \ldots, n,$$

where $K_{f,T,J}$ is the constrained Poisson kernel associated with $f$, $T$, and $J$. Note that, due to relation (4.2), we have

$$K_{f,T}^*(p(M_{Z_1}, \ldots, M_{Z_n}) \otimes I_{\mathcal{D}_{f,T}}) = p(T_1, \ldots, T_n)K_{f,T}^* \tag{4.3}$$

for any polynomial $p$ in $M_{Z_1}, \ldots, M_{Z_n}$. If $\varphi(S_1, \ldots, S_n) := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha S_\alpha$ is in $F_n^\infty$, then, for any $0 < r < 1$, $\varphi_r(S_1, \ldots, S_n) := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} a_\alpha S_\alpha$ is in the noncommutative disc algebra $\mathcal{A}_n$. Consequently,

$$\lim_{m\to\infty} \sum_{k=0}^{m} \sum_{|\alpha|=k} r^{|\alpha|} a_\alpha [f(M_Z)]_\alpha = \varphi_r(f(M_Z))$$

in the norm topology and, using relation (4.3) we obtain

$$K_{f,T}^*[\varphi_r(f(M_Z)) \otimes I_{\mathcal{D}_{f,T}}] = \varphi_r(f(T))K_{f,T}^*$$

for any $\varphi(S_1, \ldots, S_n) \in F_n^\infty$ and $0 < r < 1$. Since $T \in \mathbb{B}_f^{cnc}(\mathcal{H})$ and $M_Z := (M_{Z_1}, \ldots, M_{Z_n}) \in \mathbb{B}^{pure}(\mathbb{H}^2(f))$, we can use the $F_n^\infty$-functional calculus for row contractions. We recall that the map $A \mapsto A \otimes I$ is SOT-continuous on bounded sets of $B(F^2(H_n))$ and, due to the noncommutative von Neumann inequality [12], we have $\|\varphi_r(f(M_Z))\| \leq \|\varphi(S_1, \ldots, S_n)\|$. Therefore, we can take $r \to 1$ in the equality above and obtain

$$K_{f,T}^*(\varphi(f(M_Z)) \otimes I_{\mathcal{D}_{f,T}}) = \varphi(f(T))K_{f,T}^*$$

for any $\varphi(S_1, \ldots, S_n) \in F_n^\infty$. Consequently, we have

$$K_{f,T}^*(\psi \otimes I_{\mathcal{D}_{f,T}}) = \psi(T)K_{f,T}^*, \qquad \psi \in H^\infty(\mathbb{B}_f), \tag{4.4}$$

which implies

$$\left\langle(\psi^* \otimes I_{\mathcal{D}_{f,T}})K_{f,T}h, 1 \otimes k\right\rangle = \langle K_{f,T}\psi(T)^*h, 1 \otimes k\rangle$$

for any $\psi \in H^\infty(\mathbb{B}_f)$, $h \in \mathcal{H}$, and $k \in \mathcal{D}_{f,T}$. Hence, and using the fact that $\psi(T) = 0$ for $\psi \in J$, we obtain $\langle K_{f,T}h, \psi(1) \otimes k\rangle = 0$ for any $h \in \mathcal{H}$ and $k \in \mathcal{D}_{f,T}$. Taking into account the definition of $\mathcal{M}_{f,J}$, we deduce that $K_{f,T}(\mathcal{H}) \subseteq \mathcal{N}_{f,J} \otimes \mathcal{D}_{f,T}$. This shows that the constrained Poisson kernel $K_{f,T,J}$ satisfies the relation

$$K_{f,T,J}h = \left(P_{\mathcal{N}_{f,J}} \otimes I_{\mathcal{D}_{f,T}}\right) K_{f,T}h = K_{f,T}h, \qquad h \in \mathcal{H}. \tag{4.5}$$

Since $J$ is a left ideal of $H^\infty(\mathbb{B}_f)$, $\mathcal{N}_{f,J}$ is an invariant subspace under each operator $M_{Z_1}^*, \ldots, M_{Z_n}^*$ and therefore $B_\alpha = P_{\mathcal{N}_{f,J}} M_{Z_\alpha}|\mathcal{N}_{f,J}$ for $\alpha \in \mathbb{F}_n^+$. Since $(B_1, \ldots, B_n) \in \mathbb{B}^{pure}(\mathcal{N}_{f,J})$, we can use the $H^\infty(\mathbb{B}_f)$-functional calculus to deduce that

$$\chi(B_1, \ldots, B_n) = P_{\mathcal{N}_{f,J}}\chi(M_{Z_1}, \ldots, M_{Z_n})|_{\mathcal{N}_{f,J}} \tag{4.6}$$

for any $\chi \in H^\infty(\mathbb{B}_f)$. Taking into account relations (4.4), (4.5), and (4.6), we obtain

$$\begin{aligned}
K_{f,T,J}\chi(T_1, \ldots, T_n)^* &= \left(P_{\mathcal{N}_{f,J}} \otimes I_{\mathcal{D}_{f,T}}\right)[\chi(M_{Z_1}, \ldots, M_{Z_n})^* \otimes I_{\mathcal{D}_{f,T}}]\left(P_{\mathcal{N}_{f,J}} \otimes I_{\mathcal{D}_{f,T}}\right) K_{f,T} \\
&= \left[\left(P_{\mathcal{N}_{f,J}}\chi(M_{Z_1}, \ldots, M_{Z_n})|\mathcal{N}_{f,J}\right)^* \otimes I_{\mathcal{D}_{f,T}}\right] K_{f,T,J} \\
&= \left[\chi(B_1, \ldots, B_n)^* \otimes I_{\mathcal{D}_{f,T}}\right] K_{f,T,J}.
\end{aligned}$$

Therefore, we have

$$K_{f,T,J}\chi(T_1, \ldots, T_n)^* = \left[\chi(B_1, \ldots, B_n)^* \otimes I_{\mathcal{D}_{f,T}}\right] K_{f,T}$$

for any $\chi(B_1,\dots,B_n)\in H^\infty(\mathcal{V}_{f,J})$. In particular, we have $K_{f,T,J}T_i^*=(B_i^*\otimes I_{\mathcal{D}_{f,T}})K_{f,T,J}$ for $i=1,\dots,n$, which proves the first part of the proposition.

Due to relation (4.5) and the definition of the Poisson kernel, we have

$$\begin{aligned}\left\langle K_{f,T,J}^*K_{f,T,J}h,h\right\rangle=\|K_{f,T}h\|^2&=\lim_{q\to\infty}\left\|\sum_{\alpha\in\mathbb{F}_n^+,|\alpha|\le q}f_\alpha\otimes\Delta_{f,T}[f(T)]_\alpha^*h\right\|^2_{\mathbb{H}^2(f)\otimes\mathcal{H}}\\&=\lim_{q\to\infty}\sum_{\alpha\in\mathbb{F}_n^+,|\alpha|\le q}\left\langle[f(T)]_\alpha\Delta_{f,T}^2[f(T)]_\alpha^*h,h\right\rangle\\&=\|h\|-\lim_{q\to\infty}\left\langle\left(\sum_{\alpha\in\mathbb{F}_n^+,|\alpha|=q}[f(T)]_\alpha[f(T)]_\alpha^*\right)h,h\right\rangle\end{aligned}$$

for any $h\in\mathcal{H}$. Since $[f_1(T),\dots,f_n(T)]$ is a row contraction, the latter limit exists. The proof is complete. ☐

**Corollary 4.2.** *If $T=(T_1,\dots,T_n)$ is a pure $n$-tuple of operators in $\mathcal{V}_{f,J}^{cnc}(\mathcal{H})$, then*

$$T_\alpha T_\beta^*=K_{f,T,J}^*[B_\alpha B_\beta^*)\otimes I]K_{f,T,J},\qquad \alpha,\beta\in\mathbb{F}_n^+,$$

*and*

$$\left\|\sum_{i=1}^m q_i(T_1,\dots,T_n)q_i(T_1,\dots,T_n)^*\right\|\le\left\|\sum_{i=1}^m q_i(B_1,\dots,B_n)q_i(B_1,\dots,B_n)\right\|$$

*for any $q_i\in\mathbb{C}[Z_1,\dots,Z_n]$ and $m\in\mathbb{N}$. Moreover,*

$$\|\chi(T_1,\dots,T_n)\|\le\|\chi(B_1,\dots,B_n)\|$$

*for any $\chi\in H^\infty(\mathbb{B}_f)$.*

Let $J$ be a WOT-closed two-sided ideal of the Hardy algebra $H^\infty(\mathbb{B}_f)$. We define the *constrained characteristic function* associated with an $n$-tuple $T:=(T_1,\dots,T_n)\in\mathcal{V}_{f,J}^{cnc}(\mathcal{H})$ to be the multi-analytic operator with respect to $B_1,\dots,B_n$,

$$\Theta_{f,T,J}(W_1,\dots,W_n):\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T^*}\to\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T},$$

defined by the formal Fourier representation

$$-I_{\mathcal{N}_{f,J}}\otimes f(T)+\left(I_{\mathcal{N}_{f,J}}\otimes\Delta_{f,T}\right)\left(I_{\mathcal{N}_{f,J}\otimes\mathcal{H}}-\sum_{i=1}^n W_i\otimes f_i(T)^*\right)^{-1}[W_1\otimes I_{\mathcal{H}},\dots,W_n\otimes I_{\mathcal{H}}]\left(I_{\mathcal{N}_{f,J}}\otimes\Delta_{f,T^*}\right),$$

where $W_i:=P_{\mathcal{N}_{f,J}}\Lambda_i|_{\mathcal{N}_{f,J}}$, $i=1,\dots,n$, and $\Lambda_1,\dots,\Lambda_n$ are the right multiplication operators by the power series $f_i$ on the Hardy space $\mathbb{H}^2(f)$.

**Theorem 4.3.** *Let $f=(f_1,\dots,f_n)$ be an $n$-tuple of formal power series with the model property and let $J\ne H^\infty(\mathbb{B}_f)$ be a WOT-closed two-sided ideal of $H^\infty(\mathbb{B}_f)$. If $T:=(T_1,\dots,T_n)$ is an $n$-tuple of operators in the noncommutative variety $\mathcal{V}_{f,J}^{cnc}(\mathcal{H})$, then $T$ is unitarily equivalent to the $n$-tuple $\mathbf{T}:=(\mathbf{T}_1,\dots,\mathbf{T}_n)$ in $\mathcal{V}_{f,J}^{cnc}(\mathbf{H})$ where:*

(i) *the Hilbert space $\mathbf{H}$ is defined by*

$$\mathbf{H}:=\left[(\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T})\oplus\overline{\Delta_{\Theta_{f,T,J}}(\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T^*})}\right]\ominus\left\{\Theta_{f,T,J}f\oplus\Delta_{\Theta_{f,T,J}}f:\ f\in\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T^*}\right\},$$

*where $\Delta_{\Theta_{f,T,J}}:=\left(I-\Theta_{f,T,J}^*\Theta_{f,T,J}\right)^{1/2}$;*

(ii) *each operator $\mathbf{T}_i$, $i=1,\dots,n$, is uniquely defined by the relation*

$$\left(P_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}|_{\mathbf{H}}\right)\mathbf{T}_i^*x=(B_i^*\otimes I_{\mathcal{D}_{f,T}})\left(P_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}|_{\mathbf{H}}\right)x,\qquad x\in\mathbf{H},$$

*where $P_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}|_{\mathbf{H}}$ is a one-to-one operator, $P_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}$ is the orthogonal projection of the Hilbert space $(\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T})\oplus\overline{\Delta_{\Theta_{f,T,J}}(\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T^*})}$ onto the subspace $\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}$, and $B_i:=P_{\mathcal{N}_{f,J}}M_{Z_i}|_{\mathcal{N}_{f,J}}$ for any $i=1,\dots,n$.*

*Moreover, $T$ is in $\mathcal{V}_{f,J}^{pure}(\mathcal{H})$ if and only if the constrained characteristic function $\Theta_{f,T,J}$ is a partial isometry. In this case, $T$ is unitarily equivalent to the $n$-tuple*

$$\left(P_{\mathbf{H}}(B_1\otimes I_{\mathcal{D}_{f,T}})|_{\mathbf{H}},\ldots,P_{\mathbf{H}}(B_n\otimes I_{\mathcal{D}_{f,T}})|_{\mathbf{H}}\right),$$

*where $P_{\mathbf{H}}$ is the orthogonal projection of $\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}$ onto the Hilbert space*

$$\mathbf{H}=(\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T})\ominus\Theta_{f,T,J}(\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T^*}).$$

*Proof.* Taking into account that $\mathcal{N}_{f,J}$ is a co-invariant subspace under $\Lambda_1,\ldots,\Lambda_n$, we can see that

$$\Theta_T(\Lambda_1,\ldots,\Lambda_n)^*(\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T})\subseteq\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T^*}\quad\text{and}$$
$$P_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}\Theta_{f,T}(\Lambda_1,\ldots,\Lambda_n)|\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T^*}=\Theta_{f,T,J}(W_1,\ldots,W_n).$$

Hence, using relation $K_{f,T}K_{f,T}^*+\Theta_{f,T}\Theta_{f,T}^*=I_{\mathbb{H}^2(f)\otimes\mathcal{D}_{f,T}}$ (see [27]), the fact that range $K_{f,T}\subseteq\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}$ and $W_i^*=\Lambda_i^*|_{\mathcal{N}_{f,J}},\ i=1,\ldots,n$, we deduce that

$$I_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}-\Theta_{f,T,J}\Theta_{f,T,J}^*=K_{f,T,J}K_{f,T,J}^*,\tag{4.7}$$

where $\Theta_{f,T,J}$ is the constrained characteristic function of $f$, $T$ and $J$, and $K_{f,T,J}$ is the corresponding constrained Poisson kernel.

Now, we introduce the Hilbert space

$$\mathbf{K}:=(\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T})\oplus\overline{\Delta_{\Theta_{f,T,J}}(\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T^*})}$$

and define the operator $\Phi:\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T^*}\to\mathbf{K}$ by setting

$$\Phi x:=\Theta_{f,T,J}x\oplus\Delta_{\Theta_{f,T,J}}x,\qquad x\in\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T^*}.\tag{4.8}$$

It is easy to see that $\Phi$ is an isometry and

$$\Phi^*(y\oplus 0)=\Theta_{f,T,J}^*y,\qquad y\in\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}.\tag{4.9}$$

Hence, letting $P_{\mathbf{H}}$ be the orthogonal projection of $\mathbf{K}$ onto the subspace $\mathbf{H}$, we have

$$\|y\|^2=\|P_{\mathbf{H}}(y\oplus 0)\|^2+\|\Phi\Phi^*(y\oplus 0)\|=\|P_{\mathbf{H}}(y\oplus 0)\|^2+\|\Theta_{f,T,J}^*y\|^2$$

for any $y\in\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}$. Note also that relation (4.7) implies

$$\|K_{f,T,J}^*y\|^2+\|\Theta_{f,T,J}^*y\|^2=\|y\|^2,\qquad y\in\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}.$$

Consequently, we deduce that

$$\|K_{f,T,J}y\|=\|P_{\mathbf{H}}(y\oplus 0)\|,\qquad y\in\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}.\tag{4.10}$$

Now, we prove that $K_{f,T,J}$ is a one-to-one operator. Indeed, due to Proposition 4.1, for any $h\in\mathcal{H}$, we have

$$\|K_{f,T,J}h\|^2=\|h\|^2-\lim_{q\to\infty}\sum_{\alpha\in\mathbb{F}_n,\,|\alpha|=q}\|[f(T)]_\alpha^*h\|^2.$$

Consequently, if $K_{f,T,J}h=0$, then $\|h\|^2=\lim_{q\to\infty}\sum_{\alpha\in\mathbb{F}_n,\,|\alpha|=q}\|[f(T)]_\alpha^*h\|^2$. Since $[f_1(T),\ldots,f_n(T)]$ is a row contraction, the latter relation implies $\|h\|^2=\sum_{\alpha\in\mathbb{F}_n,\,|\alpha|=q}\|[f(T)]_\alpha^*h\|^2$ for any $q\in\mathbb{N}$. Since $T\in\mathbb{B}_f^{cnc}(\mathcal{H})$, we deduce that $h=0$, which proves that $K_{f,T,J}$ is a one-to-one operator and, therefore, the range of $K_{f,T,J}^*$ is dense in $\mathcal{H}$.

Next, we show that

$$\mathbf{H}=\{P_{\mathbf{H}}(y\oplus 0):\ y\in\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}\}^-.\tag{4.11}$$

Let $x\in\mathbf{H}$ and assume that $x\perp P_{\mathbf{H}}(y\oplus 0)$ for any $y\in\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}$. Using the definition of $\mathbf{H}$ and the fact that $\mathbf{K}$ is the closed span of all the vectors $y\oplus 0$ for $y\in\mathcal{N}_{f,J}\otimes\mathcal{D}_T$ and $\Theta_{J,T}x\oplus\Delta_{\Theta_{f,T,J}}x$ for $x\in\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T^*}$, we deduce that $x=0$. Therefore, relation (4.11) holds.

Note that, due to relation (4.10) and (4.11), there is a unique unitary operator $\Gamma:\mathcal{H}\to\mathbf{H}$ such that

$$\Gamma(K_{f,T,J}^*y)=P_{\mathbf{H}}(y\oplus 0),\qquad y\in\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}.\tag{4.12}$$

Using relations (4.7) and (4.9), and the fact that $\Phi$ is an isometry defined by (4.8), we have

$$\begin{aligned}P_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}\Gamma K_{f,T,J}^*y &= P_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}P_{\mathbf{H}}(y\oplus 0) = y - P_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}\Phi\Phi^*(y\oplus 0)\\ &= y-\Theta_{f,T,J}\Theta_{f,T,J}^*y = K_{f,T,J}K_{f,T,J}^*y\end{aligned}$$

for any $y\in\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}$. Hence, and using the fact that the range of $K_{f,T,J}^*$ is dense in $\mathcal{H}$, we obtain relation

$$P_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}\Gamma = K_{f,T,J}. \tag{4.13}$$

Now, we define $\mathbf{T}_i:\mathbf{H}\to\mathbf{H}$ be the transform of $T_i$ under the unitary operator $\Gamma:\mathcal{H}\to\mathbf{H}$ defined by (4.12). More precisely, we set $\mathbf{T}_i:=\Gamma T_i\Gamma^*$, $i=1,\ldots,n$. Since $K_{f,T,J}$ is one-to-one, relation (4.13) implies that

$$P_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}|_{\mathbf{H}} = K_{f,T,J}\Gamma^*$$

is a one-to-one operator acting from $\mathbf{H}$ to $\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}$. Due to relation (4.13) and Proposition 4.1, we obtain

$$\begin{aligned}\left(P_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}|_{\mathbf{H}}\right)\mathbf{T}_i^*\Gamma h &= \left(P_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}|_{\mathbf{H}}\right)\Gamma T_i^*h = K_{f,T,J}T_i^*h\\ &= \left(B_i^*\otimes I_{\mathcal{D}_{f,T}}\right)K_{f,T,J}h = \left(B_i^*\otimes I_{\mathcal{D}_{f,T}}\right)\left(P_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}|_{\mathbf{H}}\right)\Gamma h\end{aligned}$$

for any $h\in\mathcal{H}$. Consequently,

$$\left(P_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}|_{\mathbf{H}}\right)\mathbf{T}_i^*x = (B_i^*\otimes I_{\mathcal{D}_{f,T}})\left(P_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}|_{\mathbf{H}}\right)x,\qquad x\in\mathbf{H}. \tag{4.14}$$

Due to the fact that the operator $P_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}|_{\mathbf{H}}$ is one-to-one, the relation (4.14) uniquely determines the operators $\mathbf{T}_i^*$ for $i=1,\ldots,n$.

Now, we assume that $T:=(T_1,\ldots,T_n)\in\mathcal{V}_{f,J}^{pure}(\mathcal{H})$. According to Proposition 4.1, the constrained Poisson kernel $K_{f,T,J}:\mathcal{H}\to\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}$ is an isometry and, therefore, $K_{f,T,J}K_{f,T,J}^*$ is the orthogonal projection of $\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}$ onto $K_{f,T,J}\mathcal{H}$. Relation (4.7) implies that $K_{f,T,J}K_{f,T,J}^*$ and $\Theta_{f,T,J}\Theta_{f,T,J}^*$ are mutually orthogonal projections such that

$$K_{f,T,J}K_{f,T,J}^*+\Theta_{f,T,J}\Theta_{f,T,J}^* = I_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}.$$

This shows that $\Theta_{f,T,J}$ is a partial isometry and $\Theta_{f,T,J}^*\Theta_{f,T,J}$ is a projection. Consequently, $\Delta_{\Theta_{f,T,J}}$ is the projection on the orthogonal complement of the range of $\Theta_{f,T,J}^*$.

We remark that a vector $u\oplus\Delta_{\Theta_{f,T,J}}v\in\mathbf{K}$ is in $\mathbf{H}$ if and only if

$$\left\langle u\oplus\Delta_{\Theta_{f,T,J}}v, \Theta_{f,T,J}x\oplus\Delta_{\Theta_{f,T,J}}x\right\rangle = 0 \quad \text{ for any } \ x\in\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T^*},$$

which is equivalent to

$$\Theta_{f,T,J}^*u+\Delta_{\Theta_{f,T,J}}^2v=0. \tag{4.15}$$

Since $\Theta_{f,T,J}^*u\perp\Delta_{\Theta_{f,T,J}}^2v$, relation (4.15) holds if and only if $\Theta_{f,T,J}^*u=0$ and $\Delta_{\Theta_{f,T,J}}v=0$. Consequently, we have

$$\mathbf{H} = (\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T})\ominus\Theta_{f,T,J}(\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T^*}).$$

Note that $P_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}|_{\mathbf{H}}$ is the restriction operator and relation (4.14) implies $\mathbf{T}_i = P_{\mathbf{H}}(B_i\otimes I_{\mathcal{D}_{f,T}})|_{\mathbf{H}}$ for $i=1,\ldots,n$.

Conversely, if $\Theta_{f,T,J}$ is is a partial isometry, relation (4.7) implies that $K_{f,T,J}$ is a partial isometry. On the other hand, since $T$ is c.n.c., Proposition 4.1 implies

$$\text{SOT-}\lim_{k\to\infty}\sum_{|\alpha|=k}[f(T)]_\alpha[f(T)]_\alpha^*=0,$$

which proves that $T\in\mathcal{V}_{f,J}^{pure}(\mathcal{H})$. This completes the proof. $\square$

We remark that, if $T := (T_1, \ldots, T_n) \in \mathcal{V}^{cnc}_{f,J}(\mathcal{H})$, then $\Theta_{f,T,J}$ has dense range if and only if there is no element $h \in \mathcal{H}$, $h \neq 0$, such that

$$\lim_{k\to\infty} \sum_{|\alpha|=k} [f(T)]_\alpha [f(T)]_\alpha^* h = 0.$$

Indeed, due to Proposition 4.1, the condition above is equivalent to $\ker \left(I - K^*_{f,T,J} K_{f,T,J}\right) = \{0\}$. Using relation (4.7) , we deduce that the latter equality is equivalent to

$$\ker \Theta_{f,T,J}\Theta^*_{f,T,J} = \ker \left(I - K_{f,T,J}K^*_{f,T,J}\right) = \{0\},$$

which implies that $\Theta_{f,T,J}$ has dense range.

**Proposition 4.4.** *Let $f = (f_1, \ldots, f_n)$ be an $n$-tuple of formal power series with the model property and let $J \neq H^\infty(\mathbb{B}_f)$ be a WOT-closed two-sided ideal of $H^\infty(\mathbb{B}_f)$ such that $1 \in \mathcal{N}_{f,J}$. Then $T := (T_1, \ldots, T_n) \in \mathcal{V}^{cnc}_{f,J}(\mathcal{H})$ is unitarily equivalent to the universal $n$-tuple $(B_1 \otimes I_\mathcal{K}, \ldots, B_n \otimes I_\mathcal{K})$ for some Hilbert space $\mathcal{K}$ if and only if $\Theta_{f,T,J} = 0$.*

*Proof.* First, we assume that $T = (B_1 \otimes I_\mathcal{K}, \ldots, B_n \otimes I_\mathcal{K})$ and prove that $K_{f,T,J}F = F$ for $F \in \mathcal{N}_{f,J} \otimes \mathcal{K}$. Since $1 \in \mathcal{N}_{f,J}$, a straightforward calculation shows that $\Delta_{f,T} = P_\mathcal{K}|_{\mathcal{N}_{f,J}\otimes\mathcal{K}}$ as an operator acting on $\mathcal{N}_{f,J} \otimes \mathcal{K}$. Indeed, note that

$$\begin{aligned}
\Delta^2_{f,T} &= I_{\mathcal{N}_{f,J}\otimes\mathcal{K}} - \sum_{i=1}^n f_i(B) f_i(B)^* \otimes I_\mathcal{K} \\
&= P_{\mathcal{N}_{f,J}\otimes\mathcal{K}} \left( I_{\mathbb{H}^2(f)\otimes\mathcal{K}} - \sum_{i=1}^n f_i(M_Z) f_i(M_Z)^* \otimes I_\mathcal{K} \right) |_{\mathcal{N}_{f,J}\otimes\mathcal{K}} \\
&= P_{\mathcal{N}_{f,J}\otimes\mathcal{K}} \left( I_{\mathbb{H}^2(f)\otimes\mathcal{K}} - \sum_{i=1}^n M_{f_i} M^*_{f_i} \otimes I_\mathcal{K} \right) |_{\mathcal{N}_{f,J}\otimes\mathcal{K}} \\
&= P_\mathcal{K}|_{\mathcal{N}_{f,J}\otimes\mathcal{K}}.
\end{aligned}$$

Here we used the natural identification of $1 \otimes \mathcal{K}$ with $\mathcal{K}$. Since $\mathcal{D}_{f,T} = \mathcal{K}$, using the definition of the constrained Poisson kernel $K_{f,T,J}$, for any $F = \sum_{\beta\in\mathbb{F}_n^+} f_\beta \otimes k_\beta$ in $\mathcal{N}_{f,J} \otimes \mathcal{K} \subseteq \mathbb{H}^2(f) \otimes \mathcal{K}$, we have

$$\begin{aligned}
K_{f,T,J}F &= \sum_{\alpha\in\mathbb{F}_n^+} P_{\mathcal{N}_{f,J}} f_\alpha \otimes P_\mathcal{K}([f(B)]^*_\alpha \otimes I_\mathcal{K})F = \sum_{\alpha\in\mathbb{F}_n^+} P_{\mathcal{N}_{f,J}} f_\alpha \otimes P_\mathcal{K}([f(M_Z)]^*_\alpha \otimes I_\mathcal{K})F \\
&= \sum_{\alpha\in\mathbb{F}_n^+} P_{\mathcal{N}_{f,J}} f_\alpha \otimes P_\mathcal{K}(M^*_{f_\alpha} \otimes I_\mathcal{K})F = \sum_{\alpha\in\mathbb{F}_n^+} P_{\mathcal{N}_J} f_\alpha \otimes k_\alpha \\
&= P_{\mathcal{N}_{f,J}\otimes\mathcal{K}} F = F.
\end{aligned}$$

Due to relation (4.7) we have $\Theta_{f,T,J} = 0$. Conversely, if $\Theta_{f,T,J} = 0$, then Theorem 4.3 shows that $T$ is unitarily equivalent to $(B_1 \otimes I_{\mathcal{D}_{f,T}}, \ldots, B_n \otimes I_{\mathcal{D}_{f,T}})$. This completes the proof. ☐

Let $\Phi : \mathcal{N}_{f,J} \otimes \mathcal{K}_1 \to \mathcal{N}_{f,J} \otimes \mathcal{K}_2$ and $\Phi' : \mathcal{N}_{f,J} \otimes \mathcal{K}'_1 \to \mathcal{N}_{f,J} \otimes \mathcal{K}'_2$ be two multi-analytic operators with respect to $B_1, \ldots, B_n$, i.e., $\Phi(B_i \otimes I_{\mathcal{K}_1}) = (B_i \otimes I_{\mathcal{K}_2})\Phi$ and $\Phi'(B_i \otimes I_{\mathcal{K}'_1}) = (B_i \otimes I_{\mathcal{K}'_2})\Phi'$ for any $i = 1, \ldots, n$. We say that $\Phi$ and $\Phi'$ coincide if there are two unitary operators $\tau_j \in B(\mathcal{K}_j, \mathcal{K}'_j)$, $j = 1, 2$, such that

$$\Phi'(I_{\mathcal{N}_{f,J}} \otimes \tau_1) = (I_{\mathcal{N}_{f,J}} \otimes \tau_2)\Phi.$$

The next result shows that the constrained characteristic function is a complete unitary invariant for the $n$-tuples of operators in the noncommutative variety $\mathcal{V}^{cnc}_{f,J}(\mathcal{H})$.

**Theorem 4.5.** *Let $f = (f_1, \ldots, f_n)$ be an $n$-tuple of formal power series with the model property and let $J \neq H^\infty(\mathbb{B}_f)$ be a WOT-closed two-sided ideal of $H^\infty(\mathbb{B}_f)$. If $T := (T_1, \ldots, T_n) \in \mathcal{V}^{cnc}_{f,J}(\mathcal{H})$ and $T' := (T'_1, \ldots, T'_n) \in \mathcal{V}^{cnc}_{f,J}(\mathcal{H}')$, then $T$ and $T'$ are unitarily equivalent if and only if their constrained characteristic functions $\Theta_{f,T,J}$ and $\Theta_{f,T',J}$ coincide.*

*Proof.* Let $W:\mathcal{H}\to\mathcal{H}'$ be a unitary operator such that $T_i=W^*T_i'W$ for any $i=1,\dots,n$. Since $T\in\mathcal{C}_f^{SOT}(\mathcal{H})$ or $T\in\mathcal{C}_f^{rad}(\mathcal{H})$ and similar relations hold for $T'$, it is easy to see that

$$W\Delta_{f,T}=\Delta_{f,T'}W \quad \text{and} \quad (\oplus_{i=1}^n W)\Delta_{f,T^*}=\Delta_{f,T'^*}(\oplus_{i=1}^n W).$$

We introduce the unitary operators $\tau$ and $\tau'$ by setting

$$\tau:=W|_{\mathcal{D}_{f,T}}:\mathcal{D}_{f,T}\to\mathcal{D}_{f,T'} \quad \text{and} \quad \tau':=(\oplus_{i=1}^n W)|_{\mathcal{D}_{f,T^*}}:\mathcal{D}_{f,T^*}\to\mathcal{D}_{f,T'^*}.$$

It is easy to see that

$$(I_{\mathcal{N}_{f,J}}\otimes\tau)\Theta_{f,T,J}=\Theta_{f,T',J}(I_{\mathcal{N}_{f,J}}\otimes\tau').$$

Conversely, assume that the constrained characteristic functions of $T$ and $T'$ coincide, i.e., there exist unitary operators $\tau:\mathcal{D}_{f,T}\to\mathcal{D}_{f,T'}$ and $\tau_*:\mathcal{D}_{f,T^*}\to\mathcal{D}_{f,T'^*}$ such that

$$(I_{\mathcal{N}_{f,J}}\otimes\tau)\Theta_{f,T,J}=\Theta_{f,T',J}(I_{\mathcal{N}_{f,J}}\otimes\tau_*).$$

As consequences, we obtain

$$\Delta_{\Theta_{J,T}}=\left(I_{\mathcal{N}_{f,J}}\otimes\tau_*\right)^*\Delta_{\Theta_{f,T',J}}\left(I_{\mathcal{N}_{f,J}}\otimes\tau_*\right)$$

and

$$\left(I_{\mathcal{N}_{f,J}}\otimes\tau_*\right)\overline{\Delta_{\Theta_{J,T}}(\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T^*})}=\overline{\Delta_{\Theta_{J,T'}}(\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T'^*})}.$$

Now, we define the unitary operator $U:\mathbf{K}\to\mathbf{K}'$ by setting

$$U:=(I_{\mathcal{N}_{f,J}}\otimes\tau)\oplus(I_{\mathcal{N}_{f,J}}\otimes\tau_*),$$

where the Hilbert spaces $\mathbf{K}$ and $\mathbf{K}'$ were defined in Theorem 4.3. We remark that the operator $\Phi:\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T^*}\to\mathbf{K}$, defined in the proof of the same theorem and the corresponding $\Phi'$ satisfy the relations

$$U\Phi\left(I_{\mathcal{N}_{f,J}}\otimes\tau_*\right)^*=\Phi' \quad \text{and} \quad \left(I_{\mathcal{N}_{f,J}}\otimes\tau\right)P^{\mathbf{K}}_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}U^*=P^{\mathbf{K}'}_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T'}}, \tag{4.16}$$

where $P^{\mathbf{K}}_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}$ is the orthogonal projection of $\mathbf{K}$ onto $\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}$. Note that relation (4.16) implies

$$\begin{aligned}U\mathbf{H}&=U\mathbf{K}\ominus U\Phi(\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T^*})\\&=\mathbf{K}'\ominus\Phi'(I_{\mathcal{N}_{f,J}}\otimes\tau_*)(\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T^*})\\&=\mathbf{K}'\ominus\Phi'(\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T'^*})=\mathbf{H}'.\end{aligned}$$

Consequently, the operator $U|_{\mathbf{H}}:\mathbf{H}\to\mathbf{H}'$ is unitary. Note also that

$$(B_i^*\otimes I_{\mathcal{D}_{f,T'}})(I_{\mathcal{N}_{f,J}}\otimes\tau)=(I_{\mathcal{N}_{f,J}}\otimes\tau)(B_i^*\otimes I_{\mathcal{D}_{f,T}}). \tag{4.17}$$

Let $\mathbf{T}:=(\mathbf{T}_1,\dots\mathbf{T}_n)$ and $\mathbf{T}':=(\mathbf{T}_1',\dots\mathbf{T}_n')$ be the models provided by Theorem 4.3 for the $n$-tuples $T$ and $T'$, respectively. Using the relation (4.14) for $T'$ and $T$, and relations (4.16), (4.17), we obtain

$$\begin{aligned}P^{\mathbf{K}'}_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T'}}\mathbf{T}_i'^*Ux&=(B_i^*\otimes I_{\mathcal{D}_{T'}})P^{\mathbf{K}}_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}Ux\\&=(B_i^*\otimes I_{\mathcal{D}_{f,T'}})(I_{\mathcal{N}_{f,J}}\otimes\tau)P^{\mathbf{K}}_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}x\\&=(I_{\mathcal{N}_{f,J}}\otimes\tau)(B_i^*\otimes I_{\mathcal{D}_{f,T}})P^{\mathbf{K}}_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}x\\&=(I_{\mathcal{N}_{f,J}}\otimes\tau)P^{\mathbf{K}}_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T}}\mathbf{T}_i^*x\\&=P^{\mathbf{K}'}_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T'}}U\mathbf{T}_i^*x\end{aligned}$$

for any $x\in\mathbf{H}$ and $i=1,\dots,n$. Since $P^{\mathbf{K}'}_{\mathcal{N}_{f,J}\otimes\mathcal{D}_{f,T'}}$ is a one-to-one operator (see the proof of Theorem 4.3), we deduce that

$$(U|_{\mathbf{H}})\,\mathbf{T}_i^*=\mathbf{T}_i'^*\,(U|_{\mathbf{H}}),\qquad i=1,\dots,n.$$

According to Theorem 4.3, the $n$-tuples $T$ and $T'$ are unitarily equivalent. The proof is complete. ∎

## 5. Dilation theory on noncommutative varieties

In this section, we develop a dilation theory for $n$-tuples of operators in the noncommutative variety

$$\{(T_1,\ldots,T_n)\in \mathbb{B}_f(\mathcal{H}):\ (q\circ f)(T_1,\ldots,T_n)=0,\ q\in\mathcal{P}\},$$

where $\mathcal{P}$ is a set of homogeneous noncommutative polynomials.

Let $f=(f_1,\ldots,f_n)$ be an $n$-tuple of formal power series with the model property and let $J$ be a WOT-closed two-sided ideal of $H^\infty(\mathbb{B}_f)$. We recall that the universal model $B=(B_1,\ldots,B_n)$ for the noncommutative variety $\mathcal{V}_{f,J}^{cnc}$ is defined by $B_i:=P_{\mathcal{N}_{f,J}}M_{Z_i}|_{\mathcal{N}_{f,J}}$, for $i=1,\ldots,n$, where

$$\mathcal{N}_{f,J}:=\mathbb{H}^2(f)\ominus\mathcal{M}_{f,J}\quad\text{ and }\quad \mathcal{M}_{f,J}:=\overline{J\mathbb{H}^2(f)}.$$

**Theorem 5.1.** *Let $f=(f_1,\ldots,f_n)$ be an $n$-tuple of formal power series with the model property and let $J$ be a WOT-closed two-sided ideal of $H^\infty(\mathbb{B}_f)$ such that $1\in\mathcal{N}_{f,J}$. Then the $C^*$-algebra $C^*(B_1,\ldots,B_n)$ is irreducible.*

*If $f\in\mathcal{M}^{||}$, then all the compact operators in $B(\mathcal{N}_{f,J})$ are contained in the operator space*

$$\overline{\operatorname{span}}\{B_\alpha B_\beta^*:\ \alpha,\beta\in\mathbb{F}_n^+\}.$$

*Proof.* To prove the first part of the theorem, let $\mathcal{M}\subseteq\mathcal{N}_{f,J}$ be a nonzero subspace which is jointly reducing for $B_1,\ldots,B_n$, and let $y=\sum_{\alpha\in\mathbb{F}_n^+}a_\alpha f_\alpha$ be a nonzero power series in $\mathcal{M}$. Then there is $\beta\in\mathbb{F}_n^+$ such that $a_\beta\neq 0$. Since $f=(f_1,\ldots,f_n)$ is an $n$-tuple of formal power series with the model property, we have $M_{f_i}=f_i(M_Z)$, where $M_Z:=(M_{Z_1},\ldots,M_{Z_n})$ is either in the convergence set $\mathcal{C}_f^{SOT}(\mathbb{H}^2(f))$ or $\mathcal{C}_f^{rad}(\mathbb{H}^2(f))$. Consequently, since $1\in\mathcal{N}_{f,J}$, we obtain

$$a_\beta=P_{\mathbb{C}}^{\mathcal{N}_{f,J}}[f(B)]_\beta^*y=\left(I_{\mathcal{N}_{f,J}}-\sum_{i=1}^n f_i(B)f_i(B)^*\right)[f(B)]_\beta^*y,$$

where $B=(B_1,\ldots,B_n)$. Taking into account that $\mathcal{M}$ is reducing for $B_1,\ldots,B_n$ and $a_\beta\neq0$, we deduce that $1\in\mathcal{M}$. Using again that $\mathcal{M}$ is invariant under $B_1,\ldots,B_n$, we obtain $P_{\mathcal{N}_{f,J}}\mathbb{C}[Z_1,\ldots,Z_n]\subseteq\mathcal{M}$. Since $\mathbb{C}[Z_1,\ldots,Z_n]$ is dense in $\mathbb{H}^2(f)$, we conclude that $\mathcal{M}=\mathcal{N}_{f,J}$, which shows that $C^*(B_1,\ldots,B_n)$ is irreducible.

Now, we prove the second part of the theorem. Since $\mathcal{N}_{f,J}$ is an invariant subspace under each operator $M_{Z_i}^*$, $i=1,\ldots,n$, and $(M_{Z_1},\ldots,M_{Z_n})$ is in the set of norm-convergence (or radial norm-convergence) for the $n$-tuple $f$, the operator $f_i(B)$ is in $\overline{\operatorname{span}}\{B_\alpha B_\beta^*:\ \alpha,\beta\in\mathbb{F}_n^+\}$. Taking into account that $f=(f_1,\ldots,f_n)$ is an $n$-tuple of formal power series with the model property, we have $M_{f_i}=f_i(M_{Z_1},\ldots,M_{Z_n})$. On the other hand, since $1\in\mathcal{N}_{f,J}$ the orthogonal projection of $\mathcal{N}_{f,J}$ onto the constant power series satisfies the equation

$$P_{\mathbb{C}}^{\mathcal{N}_{f,J}}=I_{\mathcal{N}_{f,J}}-\sum_{i=1}^n f_i(B)f_i(B)^*.$$

Therefore, $P_{\mathbb{C}}^{\mathcal{N}_{f,J}}$ is also in $\overline{\operatorname{span}}\{B_\alpha B_\beta^*:\ \alpha,\beta\in\mathbb{F}_n^+\}$. Let $q(B):=\sum_{|\alpha|\leq m}a_\alpha[f(B)]_\alpha$ and let $\xi:=\sum_{\beta\in\mathbb{F}_n^+}b_\beta f_\beta\in\mathcal{N}_{f,J}$. Note

$$\begin{aligned}P_{\mathbb{C}}^{\mathcal{N}_{f,J}}q(B)^*\xi&=P_{\mathbb{C}}\sum_{|\alpha|\leq m}\overline{a}_\alpha M_{f_\alpha}^*\xi=\sum_{|\alpha|\leq m}\overline{a}_\alpha b_\alpha\\&=\left\langle\xi,\sum_{|\alpha|\leq m}a_\alpha f_\alpha\right\rangle=\langle\xi,q(B)1\rangle.\end{aligned}$$

Consequently, if $r(B):=\sum_{|\gamma|\leq p}c_\gamma[f(B)]_\gamma$, then

$$r(B)P_{\mathbb{C}}q(B)^*\xi=\langle\xi,q(B)1\rangle\, r(B)1, \tag{5.1}$$

which shows that $r(B)P_{\mathbb{C}}^{\mathcal{N}_{f,J}}q(B)^*$ is a rank one operator acting on $\mathcal{N}_{f,J}$. Since the set of all vectors of the form $\sum_{|\alpha|\leq m}a_\alpha[f(B)]_\alpha 1$, where $m\in\mathbb{N}$, $a_\alpha\in\mathbb{C}$, is dense in $\mathcal{N}_{f,J}$, and using relation (5.1), we deduce that all compact operators in $B(\mathcal{N}_{f,J})$ are in $\overline{\operatorname{span}}\{B_\alpha B_\beta^*:\ \alpha,\beta\in\mathbb{F}_n^+\}$. This completes the proof. $\square$

**Proposition 5.2.** *Under the hypotheses of Theorem 5.1, if $\mathcal{H}$, $\mathcal{K}$ are Hilbert spaces, then the $n$-tuples $(B_1\otimes I_{\mathcal{H}},\dots,B_n\otimes I_{\mathcal{H}})$ and $(B_1\otimes I_{\mathcal{K}},\dots,B_n\otimes I_{\mathcal{K}})$are unitarily equivalent if and only if their multiplicities are equal, i.e.,* $\dim\mathcal{H}=\dim\mathcal{K}$.

*Proof.* Let $U:\mathcal{N}_{f,J}\otimes\mathcal{H}\to\mathcal{N}_{f,J}\otimes\mathcal{K}$ be a unitary operator such that $U(B_i\otimes I_{\mathcal{H}})=(B_i\otimes I_{\mathcal{K}})U$ for $i=1,\dots,n$. Since $U$ is unitary , we deduce that $U(B_i^*\otimes I_{\mathcal{H}})=(B_i^*\otimes I_{\mathcal{K}})U$, $i=1,\dots,n$. Since, according to Theorem 3.1, the $C^*$-algebra $C^*(B_1,\dots,B_n)$ is irreducible, we infer that $U=I_{\mathcal{N}_{f,J}}\otimes W$ for some unitary operator $W\in B(\mathcal{H},\mathcal{K})$. Therefore, $\dim\mathcal{H}=\dim\mathcal{K}$. The converse is clear. $\square$

**Theorem 5.3.** *Let $f=(f_1,\dots,f_n)$ be an $n$-tuple of formal power series with the radial approximation property, let $\mathcal{P}\subset\mathbb{C}[Z_1,\dots,Z_n]$ be a set of homogeneous polynomials, and let $B=(B_1,\dots,B_n)$ be the universal model associated with $f$ and the WOT-closed two-sided ideal $J_{\mathcal{P}\circ f}$ generated by $q(f(M_Z))$, $q\in\mathcal{P}$, in the Hardy algebra $H^\infty(\mathbb{B}_f)$. If the $n$-tuple $T=(T_1,\dots,T_n)\in\mathbb{B}_f(\mathcal{H})$ has the property that*

$$(q\circ f)(T)=0,\qquad q\in\mathcal{P},$$

*then the linear map $\Psi_{f,T,\mathcal{P}}:\overline{\operatorname{span}}\{B_\alpha B_\beta:\ \alpha,\beta\in\mathbb{F}_n^+\}\to B(\mathcal{H})$ defined by*

$$\Psi_{f,T,\mathcal{P}}(B_\alpha B_\beta):=T_\alpha T_\beta^*,\qquad \alpha,\beta\in\mathbb{F}_n^+,$$

*is completely contractive.*

*Proof.* Let $g=(g_1,\dots,g_n)$ be the inverse of $f=(f_1,\dots,f_n)$ with respect to the composition, and assume that $g_i:=\sum_{\alpha\in\mathbb{F}_n^+}a_\alpha^{(i)}Z_\alpha$, $i=1,\dots,n$. Since $f$ has the model property, the left multiplication $M_{Z_i}:\mathbb{H}^2(f)\to\mathbb{H}^2(f)$ defined by

$$M_{Z_i}\psi:=Z_i\psi,\qquad \psi\in\mathbb{H}^2(f),$$

is a bounded left multiplier of $\mathbb{H}^2(f)$ and $M_{Z_i}=U^{-1}\varphi_i(S_1,\dots,S_n)U$, where $\varphi_i(S_1,\dots,S_n)$ is in the noncommutative Hardy algebra $F_n^\infty$ and has the Fourier representation $\sum_{\alpha\in\mathbb{F}_n^+}a_\alpha^{(i)}S_\alpha$, and $U:\mathbb{H}^2(f)\to F^2(H_n)$ is the unitary operator defined by $U(f_\alpha):=e_\alpha$, $\alpha\in\mathbb{F}_n^+$.

Since $f$ has the radial approximation property, there is $\delta\in(0,1)$ such that $rf:=(rf_1,\dots,rf_n)$ has the model property for any $r\in(\delta,1]$. We remark the the Hilbert space $\mathbb{H}^2(rf)$ is in fact $\mathbb{H}^2(f)$ with the inner product defined by $\langle f_\alpha,f_\beta\rangle_{\mathbb{H}^2(rf)}:=\frac{1}{r^{|\alpha|+|\beta|}}\delta_{\alpha\beta}$, $\alpha,\beta\in\mathbb{F}_n^+$. Denote $(g_i)_{1/r}:=\sum_{\alpha\in\mathbb{F}_n^+}a_\alpha^{(i)}\frac{1}{r^{|\alpha|}}Z_\alpha$ and note that $Z_i=(g_i)_{1/r}\circ(rf)$ for $i=1,\dots,n$. Since $rf$ has the model property for $r\in(\delta,1]$, we deduce that the multiplication $M_{Z_i}^{(r)}:\mathbb{H}^2(rf)\to\mathbb{H}^2(rf)$ defined by

$$M_{Z_i}^{(r)}\psi:=Z_i\psi,\qquad \psi\in\mathbb{H}^2(rf),$$

is a bounded left multiplier of $\mathbb{H}^2(rf)$ and $M_{Z_i}^{(r)}=(U^{(r)})^{-1}\varphi_i(\frac{1}{r}S_1,\dots,\frac{1}{r}S_n)U^{(r)}$, where $\varphi_i(\frac{1}{r}S_1,\dots,\frac{1}{r}S_n)$ is in the noncommutative Hardy algebra $F_n^\infty$ and has the Fourier representation $\sum_{\alpha\in\mathbb{F}_n^+}\frac{1}{r^{|\alpha|}}a_\alpha^{(i)}S_\alpha$, and $U^{(r)}:\mathbb{H}^2(rf)\to F^2(H_n)$ is the unitary operator defined by $U^{(r)}(f_\alpha):=\frac{1}{r^{|\alpha|}}e_\alpha$, $\alpha\in\mathbb{F}_n^+$.

For each $r\in(\delta,1]$, let $J_{\mathcal{P}\circ rf}$ be the WOT-closed two-sided ideal of $H^\infty(\mathbb{B}_{rf})$ generated by the operators $q(rf(M_Z^{(r)}))$, $q\in\mathcal{P}$. We introduce the subspace $\mathcal{N}_{rf,J_{\mathcal{P}\circ rf}}:=\mathbb{H}^2(rf)\ominus\mathcal{M}_{rf,J_{\mathcal{P}\circ rf}}$, where $\mathcal{M}_{rf,J_{\mathcal{P}\circ rf}}=\overline{J_{\mathcal{P}\circ rf}\mathbb{H}^2(rf)}$, and the operators $B_i^{(r)}:=P_{\mathcal{N}_{rf,J_{\mathcal{P}\circ rf}}}M_{Z_i}^{(r)}|_{\mathcal{N}_{rf,J_{\mathcal{P}\circ rf}}}$, $i=1,\dots,n$. We denote by $J_{\mathcal{P}}$ the WOT-closed two-sided ideal of $F_n^\infty$ generated by the operators $q(S_1,\dots,S_n)$, $q\in\mathcal{P}$. We also introduce the subspace $\mathcal{N}_{J_{\mathcal{P}}}=F^2(H_n)\ominus\mathcal{M}_{J_{\mathcal{P}}}$, where $\mathcal{M}_{J_{\mathcal{P}}}:=\overline{JF^2(H_n)}$.

Our next step is to show that $\psi=\sum_{\alpha\in\mathbb{F}_n^+}c_\alpha e_\alpha$ is in $\mathcal{N}_{J_{\mathcal{P}}}$ if and only $(U^{(r)})^{-1}\psi=\sum_{\alpha\in\mathbb{F}_n^+}c_\alpha r^{|\alpha|}f_\alpha$ is in $\mathcal{N}_{rf,J_{\mathcal{P}\circ rf}}$. First, note that

$$J_{\mathcal{P}\circ rf}=J_{\mathcal{P}}\circ rf:=\{\chi(rf(M_Z)):\ \chi\in J_{\mathcal{P}}\}.$$

Due to the definition of $\mathcal{N}_{\mathcal{P}}$, one can see that $\varphi\in\mathcal{N}_{\mathcal{P}}$ if and only if $\langle\psi,\chi(S_1,\dots,S_n)1\rangle_{F^2(H_n)}=0$ for any $\chi(S_1,\dots,S_n)=\sum_{\alpha\in\mathbb{F}_n^+}a_\alpha S_\alpha\in J_{\mathcal{P}}$, which is equivalent to $\sum_{\alpha\in\mathbb{F}_n^+}c_\alpha\bar{a}_\alpha=0$. On the other hand, note

that $\sum_{\alpha\in\mathbb{F}_n^+} c_\alpha r^{|\alpha|} f_\alpha$ is in $\mathcal{N}_{rf,J_{\mathcal{P}\circ rf}}$ if and only if

$$\left\langle \sum_{\alpha\in\mathbb{F}_n^+} c_\alpha r^{|\alpha|} f_\alpha, \chi(rf(M_Z))1 \right\rangle_{\mathbb{H}^2(rf)} = 0, \qquad \chi\in J_{\mathcal{P}}.$$

Since $f$ has the model property, $M_{f_i} = f_i(M_Z)$ and, consequently, the relation above is equivalent to $\sum_{\alpha\in\mathbb{F}_n^+} c_\alpha \bar{a}_\alpha = 0$ for any $\sum_{\alpha\in\mathbb{F}_n^+} a_\alpha S_\alpha \in J_{\mathcal{P}}$, which proves our assertion. Now, it is easy to see that

$$U^{(r)}(\mathcal{M}_{rf,J_{\mathcal{P}\circ rf}}) = \mathcal{M}_{J_{\mathcal{P}}} \quad \text{and} \quad U^{(r)}(\mathcal{N}_{rf,J_{\mathcal{P}\circ rf}}) = \mathcal{N}_{J_{\mathcal{P}}}.$$

Since $M_{Z_i}^{(r)} = (U^{(r)})^{-1}\varphi_i(\frac{1}{r}S_1,\dots,\frac{1}{r}S_n)U^{(r)}$, $i=1,\dots,n$, we deduce that

$$\begin{aligned}
B_i^{(r)} &:= P_{\mathcal{N}_{rf,J_{\mathcal{P}\circ rf}}} M_{Z_i}^{(r)}|_{\mathcal{N}_{rf,J_{\mathcal{P}\circ rf}}}\\
&= P_{\mathcal{N}_{rf,J_{\mathcal{P}\circ rf}}}(U^{(r)})^{-1}\left(P_{\mathcal{N}_{J_{\mathcal{P}}}^\perp} + P_{\mathcal{N}_{J_{\mathcal{P}}}}\right)\varphi_i\left(\frac{1}{r}S_1,\dots,\frac{1}{r}S_n\right)|_{\mathcal{N}_{J_{\mathcal{P}}}}\left(U^{(r)}|_{\mathcal{N}_{rf,J_{\mathcal{P}\circ rf}}}\right)\\
&= (U^{(r)})^{-1} P_{\mathcal{N}_{J_{\mathcal{P}}}}\varphi_i\left(\frac{1}{r}S_1,\dots,\frac{1}{r}S_n\right)|_{\mathcal{N}_{J_{\mathcal{P}}}}\left(U^{(r)}|_{\mathcal{N}_{rf,J_{\mathcal{P}\circ rf}}}\right)\\
&= \left(U^{(r)}|_{\mathcal{N}_{rf,J_{\mathcal{P}\circ rf}}}\right)^{-1} P_{\mathcal{N}_{J_{\mathcal{P}}}}\varphi_i\left(\frac{1}{r}S_1,\dots,\frac{1}{r}S_n\right)|_{\mathcal{N}_{J_{\mathcal{P}}}}\left(U^{(r)}|_{\mathcal{N}_{rf,J_{\mathcal{P}\circ rf}}}\right),
\end{aligned}$$

where $U^{(r)}|_{\mathcal{N}_{rf,J_{\mathcal{P}\circ rf}}} : \mathcal{N}_{rf,J_{\mathcal{P}\circ rf}} \to \mathcal{N}_{J_{\mathcal{P}}}$ is a unitary operator for each $r\in(\delta,1]$.

Now, we assume that $T=(T_1,\dots,T_n)\in\mathbb{B}_f(\mathcal{H})$ has the property that $(q_j\circ f)(T)=0$ for $j=1,\dots,d$, and $0<r<1$. Since the $H^\infty(\mathbb{B}_f)$ functional calculus for the $n$-tuples of operators in $\mathbb{B}_f^{cnc}(\mathcal{H})$ is a homomorphism and $q\in\mathcal{P}$ is a homogenous polynomials, we have

$$(\varphi q(S_1,\dots,S_n))(rf_1(T),\dots,rf_n(T)) = r^{\deg(q)}\varphi(rf_1(T),\dots,rf_n(T))q(f_1(T),\dots,f_n(T)) = 0 \tag{5.2}$$

for any $\varphi\in F_n^\infty$ and $q\in\mathcal{P}$, where $\deg(q)$ denotes the degree of the polynomial $q$. On the other hand, $J_{\mathcal{P}\circ rf}$ is the WOT-closed two-sided ideal of $H^\infty(\mathbb{B}_{rf})$ generated by the operators $q(rf(M_Z))$, $q\in\mathcal{P}$, for each $r\in(\delta,1]$. Since the $H^\infty(\mathbb{B}_{rf})$-functional calculus for pure $n$-tuples in $\mathbb{B}_{rf}(\mathcal{H})$ is WOT-continuous, and $(T_1,\dots,T_n)\in\mathbb{B}_{rf}^{pure}(\mathcal{H})$, relation (5.2) implies $\psi(T_1,\dots,T_n)=0$ for any $\psi\in J_{\mathcal{P}\circ rf}$ and $r\in(\delta,1)$. Therefore, $(T_1,\dots,T_n)$ is in the noncommutative variety $\mathcal{V}_{rf,J_{\mathcal{P}\circ rf}}^{pure}(\mathcal{H})$. Applying Corollary 4.2 to the $n$-tuple $(T_1,\dots,T_n)$, we deduce that there is completely contractive linear map $\Phi:\overline{\text{span}}\left\{B_\alpha^{(r)}{B_\beta^{(r)}}^* :\ \alpha,\beta\in\mathbb{F}_n^+\right\}\to B(\mathcal{H})$ uniquely defined by $\Phi\left(B_\alpha^{(r)}{B_\beta^{(r)}}^*\right) := T_\alpha T_\beta^*$ for all $\alpha,\beta\in\mathbb{F}_n^+$. Hence, and using the fact that the $n$-tuple $(B_1^{(r)},\dots,B_n^{(r)})$ is unitarily equivalent to

$$\left(P_{\mathcal{N}_{J_{\mathcal{P}}}}\varphi_1\left(\frac{1}{r}S_1,\dots,\frac{1}{r}S_n\right)|_{\mathcal{N}_{J_{\mathcal{P}}}},\dots,P_{\mathcal{N}_{J_{\mathcal{P}}}}\varphi_n\left(\frac{1}{r}S_1,\dots,\frac{1}{r}S_n\right)|_{\mathcal{N}_{J_{\mathcal{P}}}}\right),$$

and $\mathcal{N}_{J_{\mathcal{P}}}$ is invariant under $S_1^*,\dots,S_n^*$, we obtain

$$\begin{aligned}
\left\|\left[\sum_{|\alpha|,|\beta|\le m} a_{\alpha\beta}^{(ij)} T_\alpha T_\beta^*\right]_{k\times k}\right\| &\le \left\|\left[\sum_{|\alpha|,|\beta|\le m} a_{\alpha\beta}^{(ij)} B_\alpha^{(r)}{B_\beta^{(r)}}^*\right]_{k\times k}\right\|\\
&\le \left\|\left[\sum_{|\alpha|,|\beta|\le m} a_{\alpha\beta}^{(ij)} P_{\mathcal{N}_{J_{\mathcal{P}}}}\varphi_\alpha\left(\frac{1}{r}S_1,\dots,\frac{1}{r}S_n\right)\varphi_\beta\left(\frac{1}{r}S_1,\dots,\frac{1}{r}S_n\right)^*|_{\mathcal{N}_{J_{\mathcal{P}}}}\right]_{k\times k}\right\|
\end{aligned}$$

for any $m,k\in\mathbb{N}$, $a_{\alpha\beta}^{(ij)}\in\mathbb{C}$, and $i,j\in\{1,\dots,k\}$. Since $f=(f_1,\dots,f_n)$ has the radial property, $\varphi_1,\dots,\varphi_n$ are free holomorphic functions on a ball $[B(\mathcal{H})^n]_\gamma$ with $\gamma>1$. Consequently, the map $(\delta,1]\ni r\mapsto\varphi_i\left(\frac{1}{r}S_1,\dots,\frac{1}{r}S_n\right)\in B(F^2(H_n))$ is continuous in the operator norm topology. Passing to the

limit as $r \to 1$ in the inequality above, we obtain

$$\left\| \left[ \sum_{|\alpha|,|\beta|\leq m} a_{\alpha\beta}^{(ij)} T_\alpha T_\beta^* \right]_{k\times k} \right\| \leq \left\| \left[ \sum_{|\alpha|,|\beta|\leq m} a_{\alpha\beta}^{(ij)} P_{\mathcal{N}_{J_\mathcal{P}}} \varphi_\alpha(S_1,\ldots,S_n)\varphi_\beta(S_1,\ldots,S_n)^* |_{\mathcal{N}_{J_\mathcal{P}}} \right]_{k\times k} \right\|. \tag{5.3}$$

Hence, using the fact (proved above) that

$$B_i := P_{\mathcal{N}_{f,J_{\mathcal{P}\circ f}}} M_{Z_i}|_{\mathcal{N}_{f,J_{\mathcal{P}\circ f}}} = \left(U^{(1)}|_{\mathcal{N}_{f,J_{\mathcal{P}\circ f}}}\right)^{-1} P_{\mathcal{N}_{J_\mathcal{P}}} \varphi_i(S_1,\ldots,S_n)|_{\mathcal{N}_{J_\mathcal{P}}} \left(U^{(1)}|_{\mathcal{N}_{f,J_{\mathcal{P}\circ f}}}\right)$$

for each $i = 1,\ldots,n$, we obtain

$$\left\| \left[ \sum_{|\alpha|,|\beta|\leq m} a_{\alpha\beta}^{(ij)} T_\alpha T_\beta^* \right]_{k\times k} \right\| \leq \left\| \left[ \sum_{|\alpha|,|\beta|\leq m} a_{\alpha\beta}^{(ij)} B_\alpha {B_\beta}^* \right]_{k\times k} \right\|,$$

which completes the proof. ☐

Let $C^*(\mathcal{Y})$ be the $C^*$-algebra generated by a set of operators $\mathcal{Y} \subset B(\mathcal{K})$ and the identity. A subspace $\mathcal{H} \subseteq \mathcal{K}$ is called $*$-cyclic for $\mathcal{Y}$ if

$$\mathcal{K} = \overline{\text{span}}\{Xh : \ X \in C^*(\mathcal{S}),\ h \in \mathcal{H}\}.$$

**Theorem 5.4.** *Let $f = (f_1,\ldots,f_n)$ be an $n$-tuple of power series in the set $\mathcal{M}_{rad} \cap \mathcal{M}^{||}$, let $\mathcal{P} \subset \mathbb{C}[Z_1,\ldots,Z_n]$ be a set of homogeneous polynomials, and let $B = (B_1,\ldots,B_n)$ be the universal model associated with $f$ and the WOT-closed two-sided ideal $J_{\mathcal{P}\circ f}$ in $H^\infty(\mathbb{B}_f)$. If $\mathcal{H}$ is a separable Hilbert space and $T = (T_1,\ldots,T_n) \in \mathbb{B}_f(\mathcal{H})$ has the property that*

$$(q\circ f)(T) = 0, \qquad q \in \mathcal{P},$$

*then there exists a separable Hilbert space $\mathcal{K}_\pi$ and a $*$-representation $\pi : C^*(B_1,\ldots,B_n) \to B(\mathcal{K}_\pi)$ which annihilates the compact operators and*

$$\sum_{i=1}^n f_i(\pi(B_1),\ldots,\pi(B_n)) f_i(\pi(B_1),\ldots,\pi(B_n))^* = I_{\mathcal{K}_\pi},$$

*such that*

(i) *$\mathcal{H}$ can be identified with a $*$-cyclic co-invariant subspace of $\tilde{\mathcal{K}} := (\mathcal{N}_{f,J_{\mathcal{P}\circ f}} \otimes \overline{\Delta_{f,T}\mathcal{H}}) \oplus \mathcal{K}_\pi$ under the operators*

$$V_i := \begin{bmatrix} B_i \otimes I_{\overline{\Delta_{f,T}\mathcal{H}}} & 0 \\ 0 & \pi(B_i) \end{bmatrix}, \quad i = 1,\ldots,n;$$

(ii) *$T_i^* = V_i^*|\mathcal{H}, \quad i = 1,\ldots,n.$*

(iii) *$V := (V_1,\ldots,V_n) \in \mathbb{B}_f(\widetilde{\mathcal{K}})$ and*

$$(q\circ f)(V) = 0, \qquad q \in \mathcal{P}.$$

*Proof.* Applying Arveson extension theorem to the map $\Psi_{f,T,\mathcal{P}}$ of Theorem 5.3, we obtain a unital completely positive linear map $\Gamma_{f,T,\mathcal{P}} : C^*(B_1,\ldots,B_n) \to B(\mathcal{H})$ such that $\Gamma_{f,T,\mathcal{P}}(B_\alpha B_\beta) := T_\alpha T_\beta^*$ for $\alpha,\beta \in \mathbb{F}_n^+$. Consider $\tilde{\pi} : C^*(B_1,\ldots,B_n) \to B(\tilde{\mathcal{K}})$ to be a minimal Stinespring dilation of $\Gamma_{f,T,\mathcal{P}}$, i.e.,

$$\Gamma_{f,T,\mathcal{P}}(X) = P_\mathcal{H}\tilde{\pi}(X)|_\mathcal{H}, \qquad X \in C^*(B_1,\ldots,B_n),$$

and $\tilde{\mathcal{K}} = \overline{\text{span}}\{\tilde{\pi}(X)h : \ X \in C^*(B_1,\ldots,B_n), h \in \mathcal{H}\}$. It is easy to see that, for each $i = 1,\ldots,n$,

$$\begin{aligned} \Gamma_{f,T,\mathcal{P}}(B_iB_i^*) &= P_\mathcal{H}\tilde{\pi}(B_i)\tilde{\pi}(B_i^*)|_\mathcal{H} \\ &= P_\mathcal{H}\tilde{\pi}(B_i)(P_\mathcal{H} + P_{\mathcal{H}^\perp})\tilde{\pi}(B_i^*)|_\mathcal{H} \\ &= \Gamma_{f,T,\mathcal{P}}(B_iB_i^*) + (P_\mathcal{H}\tilde{\pi}(B_i)|_{\mathcal{H}^\perp})(P_{\mathcal{H}^\perp}\tilde{\pi}(B_i^*)|_\mathcal{H}). \end{aligned}$$

Consequently, we have $P_{\mathcal{H}}\tilde{\pi}(B_i)|_{\mathcal{H}^\perp}=0$ and

$$\begin{aligned}\Gamma_{f,T,\mathcal{P}}(B_\alpha X)&=P_{\mathcal{H}}(\tilde{\pi}(B_\alpha)\tilde{\pi}(X))|_{\mathcal{H}}\\&=(P_{\mathcal{H}}\tilde{\pi}(B_\alpha)|_{\mathcal{H}})(P_{\mathcal{H}}\tilde{\pi}(X)|_{\mathcal{H}})\\&=\Gamma_{f,T,\mathcal{P}}(B_\alpha)\Gamma_{f,T,\mathcal{P}}(X)\end{aligned}\tag{5.4}$$

for any $X\in C^*(B_1,\dots,B_n)$ and $\alpha\in\mathbb{F}_n^+$. Note that the Hilbert space $\tilde{\mathcal{K}}$ is separable, since $\mathcal{H}$ has the same property. Relation $P_{\mathcal{H}}\tilde{\pi}(B_i)|_{\mathcal{H}^\perp}=0$ shows that $\mathcal{H}$ is an invariant subspace under each $\tilde{\pi}(B_i)^*$, $i=1,\dots,n$. Therefore,

$$\tilde{\pi}(B_i)^*|_{\mathcal{H}}=\Gamma_{f,T,\mathcal{P}}(B_i^*)=T_i^*,\qquad i=1,\dots,n.\tag{5.5}$$

Taking into account that the subspace $\mathcal{N}_{f,J_{\mathcal{P}\circ f}}$ contains the constants, we use Theorem 5.1 to conclude that all the compact operators in $B(\mathcal{N}_{f,J_{\mathcal{P}\circ f}})$ are contained in $C^*(B_1,\dots,B_n)$. We remark that one can obtain a version of Theorem 3.2 in our new setting, in a similar manner. Consequently, the representation $\tilde{\pi}$ decomposes into a direct sum $\tilde{\pi}=\pi_0\oplus\pi$ on $\tilde{\mathcal{K}}=\mathcal{K}_0\oplus\mathcal{K}_\pi$, where $\pi_0$, $\pi$ are disjoint representations of $C^*(B_1,\dots,B_n)$ on the Hilbert spaces $\mathcal{K}_0$ and $\mathcal{K}_\pi$, respectively, such that

$$\mathcal{K}_0\simeq\mathcal{N}_{f,J_{\mathcal{P}\circ f}}\otimes\mathcal{G},\quad \pi_0(X)=X\otimes I_{\mathcal{G}},\quad X\in C^*(B_1,\dots,B_n),\tag{5.6}$$

for some Hilbert space $\mathcal{G}$, and $\pi$ is a representation which annihilates the compact operators. Since $P_{\mathbb{C}}^{\mathcal{N}_{f,J_{\mathcal{P}\circ f}}}=I_{\mathcal{N}_{f,J_{\mathcal{P}}}}-\sum_{i=1}^n f_i(B)f_i(B)^*$ is a rank one projection in the $C^*$-algebra $C^*(B_1,\dots,B_n)$, we have $\sum_{i=1}^n f_i(\pi(B_1),\dots,\pi(B_n))f_i(\pi(B_1),\dots,\pi(B_n))^*=I_{\mathcal{K}_\pi}$ and $\dim\mathcal{G}=\dim(\operatorname{range}\tilde{\pi}(P_{\mathbb{C}}^{\mathcal{N}_{f,J_{\mathcal{P}\circ f}}}))$. Using the minimality of the Stinespring representation $\tilde{\pi}$, the proof of Theorem 5.1, and the fact that $P_{\mathbb{C}}^{\mathcal{N}_{f,J_{\mathcal{P}\circ f}}}B_\alpha=0$ if $|\alpha|\geq 1$, we deduce that

$$\begin{aligned}\operatorname{range}\tilde{\pi}(P_{\mathbb{C}}^{\mathcal{N}_{f,J_{\mathcal{P}\circ f}}})&=\overline{\operatorname{span}}\{\tilde{\pi}(P_{\mathbb{C}}^{\mathcal{N}_{f,J_{\mathcal{P}\circ f}}})\tilde{\pi}(X)h:\ X\in C^*(B_1,\dots,B_n),h\in\mathcal{H}\}\\&=\overline{\operatorname{span}}\{\tilde{\pi}(P_{\mathbb{C}}^{\mathcal{N}_{f,J_{\mathcal{P}\circ f}}})\tilde{\pi}(Y)h:\ Y\text{ is compact in }B(\mathcal{N}_{f,J_{\mathcal{P}\circ f}}),\ h\in\mathcal{H}\}\\&=\overline{\operatorname{span}}\{\tilde{\pi}(P_{\mathbb{C}}^{\mathcal{N}_{f,J_{\mathcal{P}\circ f}}})\tilde{\pi}(B_\alpha P_{\mathbb{C}}^{\mathcal{N}_{f,J_{\mathcal{P}\circ f}}}B_\beta^*)h:\ \alpha,\beta\in\mathbb{F}_n^+,h\in\mathcal{H}\}\\&=\overline{\operatorname{span}}\{\tilde{\pi}(P_{\mathbb{C}}^{\mathcal{N}_{f,J_{\mathcal{P}\circ f}}})\tilde{\pi}(B_\beta^*)h:\ \beta\in\mathbb{F}_n^+,h\in\mathcal{H}\}.\end{aligned}$$

Now, due to relation (5.4), we have

$$\begin{aligned}\left\langle\tilde{\pi}(P_{\mathbb{C}}^{\mathcal{N}_{f,J_{\mathcal{P}\circ f}}})\tilde{\pi}(B_\alpha^*)h,\tilde{\pi}(P_{\mathbb{C}}^{\mathcal{N}_{f,J_{\mathcal{P}\circ f}}})\tilde{\pi}(B_\beta^*)k\right\rangle&=\left\langle h,\pi(B_\alpha)\pi(P_{\mathbb{C}}^{\mathcal{N}_{f,J_{\mathcal{P}\circ f}}})\pi(B_\beta^*)h\right\rangle\\&=\left\langle h,T_\alpha\left(I_{\mathcal{H}}-\sum_{i=1}^n f_i(T)f_i(T)^*\right)T_\beta^*h\right\rangle\\&=\left\langle\Delta_{f,T}T_\alpha^*h,\Delta_{f,T}T_\beta^*k\right\rangle\end{aligned}$$

for any $h,k\in\mathcal{H}$ and $\alpha,\beta\in\mathbb{F}_n^+$. Consequently, the map $\Lambda:\operatorname{range}\tilde{\pi}(P_{\mathbb{C}}^{\mathcal{N}_{f,J_{\mathcal{P}\circ f}}})\to\overline{\Delta_{f,T}\mathcal{H}}$ defined by

$$\Lambda(\tilde{\pi}(P_{\mathbb{C}}^{\mathcal{N}_{f,J_{\mathcal{P}\circ f}}})\tilde{\pi}(B_\alpha^*)h):=\Delta_{f,T}T_\alpha^*,\quad h\in\mathcal{H},$$

can be extended by linearity and continuity to a unitary operator. Hence,

$$\dim[\operatorname{range}\pi(P_{\mathbb{C}}^{\mathcal{N}_{f,J_{\mathcal{P}\circ f}}})]=\dim\overline{\Delta_{f,T}\mathcal{H}}=\dim\mathcal{G}.$$

Under the appropriate identification of $\mathcal{G}$ with $\overline{\Delta_{f,T}\mathcal{H}}$ and using relations (5.5) and (5.6), we obtain the required dilation.

To prove item (iii), note that since $B_i:=P_{\mathcal{N}_{f,J_{\mathcal{P}\circ f}}}M_{Z_i}|_{\mathcal{N}_{f,J_{\mathcal{P}\circ f}}}$, $i=1,\dots,n$, we have $(q\circ f)(B_1,\dots,B_n)=0$, $q\in\mathcal{P}$. Taking into account that $q\in\mathcal{P}$ is a polynomial and $f\in\mathcal{M}^{||}$, the latter equality implies

$(q\circ f)(\pi(B_1),\dots,\pi(B_n))=0$. Therefore, $(q\circ f)(V_1,\dots,V_n)=0$ for $q\in\mathcal{P}$. On the other hand, since $\sum_{i=1}^n f_i(B_1,\dots,B_n)f_i(B_1,\dots,B_n)^*\le I$ and $f\in\mathcal{M}^{||}$, we also have

$$\sum_{i=1}^n f_i(\pi(B_1),\dots,\pi(B_n))f_i(\pi(B_1),\dots,\pi(B_n))^*\le I,$$

which proves that $(\pi(B_1),\dots,\pi(B_n))\in\mathbb{B}_f(\mathcal{K}_\pi)$. Consequently, $(V_1,\dots,V_n)\in\mathbb{B}_f(\widetilde{\mathcal{K}})$. This completes the proof. □

We remark that if, in addition to the hypotheses of Theorem 5.4,

$$\overline{\operatorname{span}}\,\{B_\alpha B_\beta^*:\ \alpha,\beta\in\mathbb{F}_n^+\}=C^*(B_1,\dots,B_n),$$

then the map $\Gamma_{f,T,\mathcal{P}}$ is unique and the dilation is minimal, i.e., $\tilde{\mathcal{K}}=\bigvee_{\alpha\in\mathbb{F}_n^+}V_\alpha\mathcal{H}$. In this case, the minimal dilation of Theorem 5.4 is unique, due to the uniqueness of the minimal Stinespring representation.

**Corollary 5.5.** *Let $V:=(V_1,\dots,V_n)$ be the dilation of Theorem 5.4. Then,*

(i) *$V$ is a pure $n$-tuple if and only if $T$ is pure;*
(ii) *$f_1(V)f_1(V)^*+\cdots+f_n(V)f_n(V)^*=I$ if and only if*

$$f_1(T)f_1(T)^*+\cdots+f_n(T)f_n(T)^*=I$$

*Proof.* Note that

$$\sum_{|\alpha|=k}[f(T)]_\alpha[f(T)]_\alpha^*=P_\mathcal{H}\begin{bmatrix}\sum_{|\alpha|=k}[f(B)]_\alpha[f(B)]_\alpha^*\otimes I_{\overline{\Delta_{f,T}\mathcal{H}}} & 0\\ 0 & I_{\mathcal{K}_\pi}\end{bmatrix}|_\mathcal{H}$$

and, consequently,

$$\text{SOT-}\lim_{k\to\infty}\sum_{|\alpha|=k}[f(T)]_\alpha[f(T)]_\alpha^*=P_\mathcal{H}\begin{bmatrix}0&0\\0&I_{\mathcal{K}_\pi}\end{bmatrix}|_\mathcal{H}.$$

Hence, we deduce that $T$ is a pure $n$-tuple if and only if $\mathcal{H}\perp(0\oplus\mathcal{K}_\pi)$, i.e., $\mathcal{H}\subseteq\mathcal{N}_{f,J_{\mathcal{P}\circ f}}\otimes\overline{\Delta_{f,T}\mathcal{H}}$. Taking into account that $\mathcal{N}_{f,J_{\mathcal{P}\circ f}}\otimes\overline{\Delta_{f,T}\mathcal{H}}$ is reducing for each operator $V_i$, $i=1,\dots,n$, and $\tilde{\mathcal{K}}$ is the smallest reducing subspace for the same operators, which contains $\mathcal{H}$, we draw the conclusion that $\tilde{\mathcal{K}}=\mathcal{N}_{f,J_{\mathcal{P}\circ f}}\otimes\overline{\Delta_T\mathcal{H}}$. To prove item (ii), assume that $\sum_{i=1}^n f_i(V)f_i(V)^*=I_{\tilde{\mathcal{K}}}$. Since

$$\sum_{|\alpha|=k}[f(V)]_\alpha[f(V)]_\alpha^*=\begin{bmatrix}\sum_{|\alpha|=k}[f(B)]_\alpha[f(B)]_\alpha^*\otimes I_{\overline{\Delta_{f,T}\mathcal{H}}} & 0\\ 0 & I_{\mathcal{K}_\pi}\end{bmatrix},$$

we must have $\sum_{|\alpha|=k}[f(B)]_\alpha[f(B)]_\alpha^*\otimes I_{\overline{\Delta_{f,T}\mathcal{H}}}=I_{\mathcal{K}_0}$ for any $k=1,2,\dots$. Since

$$\text{SOT-}\lim_{k\to\infty}\sum_{|\alpha|=k}[f(B)]_\alpha[f(B)]_\alpha^*=0,$$

we deduce that $\mathcal{K}_0=\{0\}$. Now, using the proof of Theorem 5.4, we concluded that $\mathcal{G}=\{0\}$ and, therefore, $\Delta_{f,T}=0$. The converse is straightforward. The proof is complete. □

## 6. Beurling type theorem and commutant lifting in noncommutative varieties

In this section, we provide a Beurling type theorem characterizing the invariant subspaces under the universal $n$-tuple associated with a noncommutative variety $\mathcal{V}_{f,J}^{pure}(\mathcal{H})$, and a commutant lifting theorem for $n$-tuples of operators in $\mathcal{V}_{f,J}^{pure}(\mathcal{H})$.

**Theorem 6.1.** *Let $f = (f_1, \ldots, f_n)$ be an $n$-tuple of power series with the model property. Let $J \neq H^\infty(\mathbb{B}_f)$ be a WOT-closed two-sided ideal of the Hardy algebra $H^\infty(\mathbb{B}_f)$ and let $(B_1, \ldots, B_n) \in \mathcal{V}_{f,J}^{cnc}(\mathcal{N}_{f,J})$ be the corresponding universal model. A subspace $\mathcal{M} \subseteq \mathcal{N}_{f,J} \otimes \mathcal{K}$ is invariant under each operator $B_1 \otimes I_\mathcal{K}, \ldots, B_n \otimes I_\mathcal{K}$ if and only if there exists a Hilbert space $\mathcal{G}$ and an operator $\Theta : \mathcal{N}_{f,J} \otimes \mathcal{G} \to \mathcal{N}_{f,J} \otimes \mathcal{K}$ with the following properties:*

(i) $\Theta$ *is a partial isometry and*

$$\Theta(B_i \otimes I_\mathcal{G}) = (B_i \otimes I_\mathcal{K})\Theta, \quad i = 1, \ldots, n.$$

(ii) $\mathcal{M} = \Theta\,(\mathcal{N}_{f,J} \otimes \mathcal{G})$.

*Proof.* First, note that the subspace $\mathcal{N}_{f,J} \otimes K$ is invariant under each operator $M_{Z_i}^* \otimes I_\mathcal{K}$, $i = 1, \ldots, n$, and

$$(M_{Z_i}^* \otimes I_\mathcal{K})|_{\mathcal{N}_{f,J} \otimes \mathcal{K}} = B_i^* \otimes I_\mathcal{K}, \quad i = 1, \ldots, n.$$

Since the subspace $[\mathcal{N}_{f,J} \otimes \mathcal{K}] \ominus \mathcal{M}$ is invariant under $B_i^* \otimes I_\mathcal{K}$, $i = 1, \ldots, n$, it is also invariant under each operator $M_{Z_i}^* \otimes I_\mathcal{K}$. Consequently, the subspace

$$\mathcal{E} := [\mathbb{H}^2(f) \otimes \mathcal{K}] \ominus \{[\mathcal{N}_{f,J} \otimes \mathcal{K}] \ominus \mathcal{M}\} = [\mathcal{M}_{f,J} \otimes \mathcal{K}] \oplus \mathcal{M} \tag{6.1}$$

is invariant under $M_{Z_i} \otimes I_\mathcal{K}$, $i = 1, \ldots, n$, where $\mathcal{M}_{f,J} := \mathbb{H}^2(f) \ominus \mathcal{N}_{f,J}$. Using the Beurling type characterization of the invariant subspaces under $M_{Z_1}, \ldots, M_{Z_n}$ (see Theorem 5.2 from [27]), we find a Hilbert space $\mathcal{G}$ and an isometric operator $\Psi : \mathbb{H}^2(f) \otimes \mathcal{G} \to \mathbb{H}^2(f) \otimes \mathcal{K}$ such that $\Psi(M_{Z_i} \otimes I_\mathcal{G}) = (M_{Z_i} \otimes I_\mathcal{K})$ for $i = 1, \ldots, n$ and

$$\mathcal{E} = \Psi[\mathbb{H}^2(f) \otimes \mathcal{G}].$$

Since $\Psi$ is an isometry, we have $P_\mathcal{E} = \Psi\Psi^*$, where $P_\mathcal{E}$ is the orthogonal projection of $\mathbb{H}^2(f) \otimes \mathcal{K}$ onto $\mathcal{E}$. Note that the subspace $\mathcal{N}_{f,J} \otimes \mathcal{K}$ is invariant under $\Psi^*$. Setting $\Theta := P_{\mathcal{N}_{f,J} \otimes \mathcal{K}} \Psi|_{\mathcal{N}_{f,J} \otimes \mathcal{G}}$, we have

$$P_{\mathcal{N}_{f,J} \otimes \mathcal{K}} P_\mathcal{E}|_{\mathcal{N}_{f,J} \otimes \mathcal{K}} = \Theta\Theta^*.$$

Hence, and using relation (6.1), we deduce that $P_\mathcal{M} = \Theta\Theta^*$, where $P_\mathcal{M}$ is the orthogonal projection of $\mathcal{N}_{f,J} \otimes \mathcal{K}$ onto $\mathcal{M}$. Therefore $\Theta$ is a partial isometry and $\mathcal{M} = \Theta\,[\mathcal{N}_{f,J} \otimes \mathcal{G}]$. The proof is complete. $\square$

We recall that, due to Theorem 4.3, any $n$-tuple $(T_1, \ldots, T_n)$ in the noncommutative variety $\mathcal{V}_{f,J}^{pure}(\mathcal{H})$ is unitarily equivalent to the compression of $[B_1 \otimes I_\mathcal{K}, \ldots, B_n \otimes I_\mathcal{K}]$ to a co-invariant subspace $\mathcal{E}$ under each operator $B_i \otimes I_\mathcal{K}$, $i = 1, \ldots, n$. Therefore, we have

$$T_i = P_\mathcal{E}(B_i \otimes I_\mathcal{K})|_\mathcal{E}, \qquad i = 1, \ldots, n.$$

The following result is a commutant lifting theorem for $n$-tuples of operators in the noncommutative variety $\mathcal{V}_{f,J}^{pure}(\mathcal{H})$.

**Theorem 6.2.** *Let $f = (f_1, \ldots, f_n)$ be an $n$-tuple of power series with the model property. Let $J \neq H^\infty(\mathbb{B}_f)$ be a WOT-closed two-sided ideal of the Hardy algebra $H^\infty(\mathbb{B}_f)$ and let $(B_1, \ldots, B_n)$ and be the corresponding universal model acting on $\mathcal{N}_{f,J}$. For each $j = 1, 2$, let $\mathcal{K}_j$ be a Hilbert space and $\mathcal{E}_j \subseteq \mathcal{N}_{f,J} \otimes \mathcal{K}_j$ be a co-invariant subspace under each operator $B_1 \otimes I_\mathcal{K}, \ldots, B_n \otimes I_\mathcal{K}$. If $X : \mathcal{E}_1 \to \mathcal{E}_2$ is a bounded operator such that*

$$X[P_{\mathcal{E}_1}(B_i \otimes I_{\mathcal{K}_1})|_{\mathcal{E}_1}] = [P_{\mathcal{E}_2}(B_i \otimes I_{\mathcal{K}_2})]|_{\mathcal{E}_2} X, \qquad i = 1, \ldots, n,$$

*then there exists an operator $G : \mathcal{N}_{f,J} \otimes \mathcal{K}_1 \to \mathcal{N}_{f,J} \otimes \mathcal{K}_2$ with the following properties:*

(i) $G(B_i \otimes I_{\mathcal{K}_1}) = (B_i \otimes I_{\mathcal{K}_2})G$ *for* $i = 1, \ldots, n$;

(ii) $G^*\mathcal{E}_2 \subseteq \mathcal{E}_1$, $G^*|\mathcal{E}_2 = X^*$, *and* $\|G\| = \|X\|$.

*Proof.* Note that the subspace $\mathcal{N}_{f,J}\otimes\mathcal{K}_j$ is invariant under each operator $M_{Z_i}^*\otimes I_{\mathcal{K}_j}$, $i=1,\dots,n$, and

$$(M_{Z_i}^*\otimes I_{\mathcal{K}_j})|_{\mathcal{N}_{f,J}\otimes\mathcal{K}_j}=B_i^*\otimes I_{\mathcal{K}_j},\quad i=1,\dots,n.$$

Since $\mathcal{E}_j\subseteq\mathcal{N}_{f,J}\otimes\mathcal{K}_j$ is invariant under $B_i^*\otimes I_{\mathcal{K}_j}$ it is also invariant under $M_{Z_i}^*\otimes I_{\mathcal{K}_j}$ and

$$(M_{Z_i}^*\otimes I_{\mathcal{K}_j})|_{\mathcal{E}_j}=(B_i^*\otimes I_{\mathcal{K}_j})|_{\mathcal{E}_j},\quad i=1,\dots,n.$$

Consequently, the intertwining relation in the hypothesis implies

$$XP_{\mathcal{E}_1}(M_{Z_i}\otimes I_{\mathcal{K}_1})|_{\mathcal{E}_1}=P_{\mathcal{E}_2}(M_{Z_i}\otimes I_{\mathcal{K}_2})|_{\mathcal{E}_2}X,\quad i=1,\dots,n.$$

We remark that, for each $j=1,2$, the $n$-tuple $(M_{Z_1}\otimes I_{\mathcal{K}_j},\dots,M_{Z_n}\otimes I_{\mathcal{K}_j})$ is a dilation of the $n$-tuple

$$[P_{\mathcal{E}_j}(M_{Z_1}\otimes I_{\mathcal{K}_j})|_{\mathcal{E}_j},\dots,P_{\mathcal{E}_j}(M_{Z_n}\otimes I_{\mathcal{K}_j})|_{\mathcal{E}_j}].$$

Applying Theorem 9.1 from [27], we find an operator $\Phi:\mathbb{H}^2(f)\otimes\mathcal{K}_1\to\mathbb{H}^2(f)\otimes\mathcal{K}_2$ with the following properties:

(i) $\Phi(M_{Z_i}\otimes I_{\mathcal{K}_1})=(M_{Z_i}\otimes I_{\mathcal{K}_2})\Phi$ for $i=1,\dots,n$;
(ii) $\Phi^*\mathcal{E}_2\subseteq\mathcal{E}_1$, $\Phi^*|\mathcal{E}_2=X^*$, and $\|\Phi\|=\|X\|$.

Set $G:=P_{\mathcal{N}_{f,J}\otimes\mathcal{K}_2}\Phi|_{\mathcal{N}_J\otimes\mathcal{K}_1}$. Since $\Phi^*(\mathcal{N}_{f,J}\otimes\mathcal{K}_2)\subseteq\mathcal{N}_{f,J}\otimes\mathcal{K}_1$, the subspace $\mathcal{N}_{f,J}\otimes\mathcal{K}_j$ is invariant under each operator $M_{Z_1}^*\otimes I_{\mathcal{K}_j},\dots,M_{Z_n}^*\otimes I_{\mathcal{K}_j}$, and $\mathcal{E}_j\subseteq\mathcal{N}_{f,J}\otimes\mathcal{K}_j$, the relations above imply $G(B_i\otimes I_{\mathcal{K}_1})=(B_i\otimes I_{\mathcal{K}_2})G$ for $i=1,\dots,n$, $G^*\mathcal{E}_2\subseteq\mathcal{E}_1$, and $G^*|_{\mathcal{E}_2}=X^*$. Hence, we deduce that $\|X\|\le\|G\|\le\|\Phi\|=\|X\|$. Therefore, $\|G\|=\|X\|$. The proof is complete. $\square$

**The commutative case.** Let $f=(f_1,\dots,f_n)$ be an $n$-tuple of power series with the model property. Let $J_c$ be the $WOT$-closed two-sided ideal of the Hardy algebra $H^\infty(\mathbb{B}_f)$ generated by the commutators

$$M_{Z_i}M_{Z_j}-M_{Z_j}M_{Z_i},\qquad i,j=1,\dots,n.$$

According to [27], the subspace $\mathcal{N}_{f,J_c}$ coincides with the symmetric Hardy space associated with $\mathbb{B}_f$, $\mathbb{H}_s^2(f)$, which can be identified with the Hilbert space $H^2(\mathbb{B}_f^<(\mathbb{C}))$ of holomorphic functions on $\mathbb{B}_f^<(\mathbb{C})$, namely, the reproducing kernel Hilbert space with reproducing kernel $K_f:\mathbb{B}_f^<(\mathbb{C})\times\mathbb{B}_f^<(\mathbb{C})\to\mathbb{C}$ defined by

$$K_f(\mu,\lambda):=\frac{1}{1-\sum_{i=1}^n f_i(\mu)\overline{f_i(\lambda)}},\qquad \lambda,\mu\in\mathbb{B}_f^<(\mathbb{C}).$$

We recall that

$$\mathbb{B}_f^<(\mathbb{C}):=\{\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n:\ \lambda=g(f(\lambda))\ \text{ and }\ \sum_{i=1}^n|f_i(\lambda)|^2<1\}=g(\mathbf{B}_n),$$

where $\mathbf{B}_n:=\{(z_1,\dots,z_n)\in\mathbb{C}^n:\ \sum_{i=1}^n|z_i|^2<1\}$ and $g=(g_1,\dots,g_n)$ is the inverse of $f$ with respect to the composition. The algebra $P_{\mathbb{H}_s^2(f)}H^\infty(\mathbb{B}_f)|_{\mathbb{H}_s^2(f)}$ coincides with the $WOT$-closed algebra generated by the operators $L_i:=P_{\mathbb{H}_s^2(f)}M_{Z_i}|_{\mathbb{H}_s^2(f)}$, $i=1,\dots,n$, and can be identified with the algebra of all multipliers of the Hilbert space $H^2(\mathbb{B}_f^<(\mathbb{C}))$. Under this identification, the operators $L_1,\dots,L_n$ become the multiplication operators $M_{z_1},\dots,M_{z_n}$ by the coordinate functions $z_1,\dots,z_n$.

Under the above-mentioned identifications, if $T:=(T_1,\dots,T_n)\in\mathbb{B}_f(\mathcal{H})$ is such that

$$T_iT_j=T_jT_i,\qquad i,j=1,\dots,n,$$

then the characteristic function of $T$ is the multiplier $M_{\Theta_{f,J_c,T}}:\mathbb{H}^2(\mathbb{B}_f^<(\mathbb{C}))\otimes\mathcal{D}_{f,T^*}\to\mathbb{H}^2(\mathbb{B}_f^<(\mathbb{C}))\otimes\mathcal{D}_{f,T}$ defined by the operator-valued analytic function

$$\Theta_{f,J_c,T}(z):=-f(T)+\Delta_{f,T}\left(I-\sum_{i=1}^n f_i(z)f_i(T)^*\right)^{-1}[f_1(z)I_{\mathcal{H}},\dots,f_n(z)I_{\mathcal{H}}]\,\Delta_{f,T^*},\qquad z\in\mathbb{B}_f^<(\mathbb{C}).$$

We remark that all the results of the last three sections can be written in this commutative setting.

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